A judgmental analysis of linear logic

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Abstract

We reexamine the foundations of linear logic, developing a system of natural deduction following Martin-Löf's separation of judgments from propositions. Our construction yields a clean and elegant formulation that accounts for a rich set of multiplicative, additive, and exponential connectives, extending dual intuitionistic linear logic but differing from both classical linear logic and Hyland and de Paiva's full intuitionistic linear logic. We also provide a corresponding sequent calculus that admits a simple proof of the admissibility of cut by a single structural induction. Finally, we show how to interpret classical linear logic (with or without the MIX rule) in our system, employing a form of double-negation translation.

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1 Introduction

Central to the design of linear logic [16] are the beautiful symmetries exhibited by the classical sequent calculus. This has led to applications in the areas of concurrent computation [24, 1] and games [2] among others, where the symmetry captures a related symmetry in the domain. However, in many situations, the asymmetry of intuitionistic natural deduction seems a better fit. For example, functional computation has an asymmetry between a function's arguments and its return value. Logic programming maintains an asymmetry between the program and the goal. Intuitionistic versions of linear logic have been used to explore interesting phenomena in functional computation (see, for example, [26, 1, 7, 41, 21, 3, 6]), logic programming [20], and logical frameworks [10].

In this paper, we analyze linear logic in an inherently asymmetric natural deduction formulation following Martin-Löf's methodology of separating judgments from propositions [27]. We require minimal judgmental notions – linear hypothetical judgments, categorical judgments, and ordinary hypothetical judgments suffice to explain a full range of intuitionistic linear propositional connectives: $\otimes$, $\mathbf{1}$, $\multimap$, $\&$, $\top$, $\oplus$, $\mathbf{0}$, $!$, $?$, $\perp$, $\neg$, and $\boxplus$, the last being a modal disjunction. The judgmental construction gives a clean and elegant proof theory, both in natural deduction and the sequent calculus. For example, we obtain a proof of cut-admissibility by a simple structural induction. We refer to the resulting system as *judgmental intuitionistic linear logic* (JILL).

As expected, the meanings of some of the connectives in JILL differ somewhat from that in classical linear logic. We do not, however, sacrifice any expressive power because we can interpret classical linear logic (with or without the MIX rule) into a fragment of JILL. We use a compositional embedding that employs a form of double-negation translation. Reformulating this translation can give a direct judgmental account of classical linear logic, showing that reasoning in classical linear logic corresponds to finding a contradication among assumptions about truth and falsehood. Similarly, reasoning in classical linear logic with MIX corresponds to finding a means to consume all resources and pay off all debts. This correspondence fills an important gap in the judgmental description of the MIX rules.

Much related work exists, so we only briefly touch upon it here. We view the judgmental reconstruction of modal and lax logic [33] as the motivation for this approach. We also owe much to Polakow's development of ordered logic [34], which employs linear and ordered hypothetical judgments, but does not introduce possibility and its associated connectives. JILL contains, as fragments, both dual intuitionistic linear logic (DILL) [5] and hereditary Harrop logic underlying linear logic programming [20]. The contribution of JILL with respect to these systems is the judgmental account, which gives rise to the new $?$, $\perp$ and $\neg$ connectives and a modal disjunction, $\boxplus$. Full intuitionistic linear logic (FILL) [23] does not have additives[1] and does not proceed via a judgmental account. It requires either proof terms [8] or occurrence labels [9] in order to formulate the rules for linear implication, which makes it difficult to understand the meanings of the connectives in isolation. On the other hand, multiplicative disjunction $⅋$ in FILL seems closer to its classical counterpart than our modal disjunction $\boxplus$; furthermore, FILL has a clear categorical semantics [11] that we have not yet explored for JILL.

We present a strucural cut-admissibility proof for a sequent calculus formulation of JILL. Related structural cut-elimination proofs have appeared for intuitionistic and classical logic [32], classical linear logic [31], and ordered logic [35], but these did not incorporate possibility and related connectives ($?$, $\perp$, $\neg$, and $\boxplus$). To our knowledge, the double-negation translation from classical into intuitionistic linear logic that can optionally account for additional structural rules such as weakening or Girard's MIX rules is also new in this paper. Lamarche has previously given a more complex double-negation translation from classical linear logic into intuitionistic linear logic using a one-sided sequent calculus with polarities [25], which is essentially a one-sided version of Girard's LU [17].

[1] These were deemed straightforward [12], though this is not obvious to the present authors.

This paper is organized as follows. In Sec. 2, we describe natural deduction for JILL in terms of the required judgmental notions. In particular, we introduce the possibility judgment (2.3), multiplicative contradiction and negation (2.4), and the modal disjunction $\boxplus$ (2.5). In Sec. 3, we derive a sequent calculus for JILL and prove a structural cut-admissibility theorem (Thm. 3). In Sec. 4, we give an interpretation of classical linear logic into JILL and further show how to modify it to give a logical justification for the classical MIX rules. In an appendix we present an alternate formulation of JILL with multiple conclusions.

2 Natural deduction for JILL

We take a foundational view of logic based on the approach laid out by Martin-Löf in his Siena lectures in 1983 [27], more recently extended to incorporate categorical judgments [33]. This view separates *judgments* from *propositions*. Evident judgments become objects of knowledge and proofs provide the requisite evidence. In logic, we concern ourselves with particular judgments such as *A is a proposition* and *A is true* (for propositions A). To understand the meaning of a proposition, we need to understand what counts as a verification of that proposition. Consequently, the inference rules characterizing truth of propositions define their meaning, as long as they satisfy certain consistency conditions that we call *local soundness* and *local completeness* [33]. We sharpen this analysis further when we introduce the sequent calculus in Sec. 3.

2.1 Linear hypothetical judgments

Before describing particular logical connectives, a brief discussion of the basic judgment forms: we focus on propositional logic, so we dispense with the formation judgment "*A is a proposition*". For propositions A, we write $A\ \ true$ to express the judgment "*A is true*". Reasoning from assumptions is fundamental in logic and captured by Martin-Löf in the form of hypothetical judgments. In linear logic, we further refine our reasoning by requiring assumptions to have *exactly one use* in a proof, which is the notion of a *linear hypothetical judgment*.

$$\underbrace{B_1\ \ true, \ldots, B_k\ \ true}_{\Delta} \Vdash C\ \ true$$

We refer to Δ as the *linear hypotheses*, and allow free exchange among the linear hypotheses while disallowing weakening and contraction. The single-use restriction on linear hypothesis in the proof of $C\ \ true$ suggests a view of linear hypotheses as *resources* used to accomplish a *goal* $C\ \ true$ – a proof thus corresponds to a plan for achieving the goal with the given resources. This interpretation yields the following hypothesis rule.

$$\frac{}{A\ \ true \Vdash A\ \ true}\ \text{hyp}$$

Dual to the hypothesis rule, we have a principle of substitution that lets one substitute proofs for uses of linear hypotheses in a derivation.

Principle 1 (Substitution).
If $\Delta \Vdash A\ \ true$ *and* $\Delta', A\ \ true \Vdash C\ \ true$, *then* $\Delta, \Delta' \Vdash C\ \ true$.

We do not realize the substitution principle as an inference rule in the logic but intend it as a structural property of the logic maintained by all other inference rules. Once the set of connectives is fixed, together with corresponding inference rules, we can prove the substitution principle as a meta-theorem by induction over the structure of derivations. Meanwhile, the substitution principle can be used to show local soundness and completeness of the inference rules characterizing the connectives.

In the spirit of the judgmental approach, one thinks of the meaning of a connective as defined by its *introduction rule*. For example, $A \otimes B$ expresses the *simultaneous truth* of both A and B. To achieve such a goal, the constituent goals must be independently established; *i.e.*, a portion of the resources must achieve A and the remaining resources must achieve B.

$$\frac{\Delta \Vdash A\ \mathit{true} \quad \Delta' \Vdash B\ \mathit{true}}{\Delta, \Delta' \Vdash A \otimes B\ \mathit{true}}\ \otimes I$$

Conversely, if one can establish $A \otimes B\ \mathit{true}$, then one may assume resources $A\ \mathit{true}$ and $B\ \mathit{true}$ available simultaneously when establishing a goal $C\ \mathit{true}$.

$$\frac{\Delta \Vdash A \otimes B\ \mathit{true} \quad \Delta', A\ \mathit{true}, B\ \mathit{true} \Vdash C\ \mathit{true}}{\Delta, \Delta' \Vdash C\ \mathit{true}}\ \otimes E$$

Local soundness ensures that the elimination rules for connectives are not too strong: given sufficient evidence for the premisses, one can find sufficient evidence for the conclusion. We show this by means of a *local reduction*, written as $\stackrel{R}{\Longrightarrow}$, reducing any proof containing an introduction immediately followed by the elimination of a connective to a proof without the connective. For $\otimes$, we use the substitution principle twice, first substituting $\mathcal{D}_1$ for uses of A and next substituting $\mathcal{D}_2$ for uses of B in $\mathcal{E}$ to yield $\mathcal{E}'$.

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta \Vdash A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Delta' \Vdash B\end{array}}{\Delta, \Delta' \Vdash A \otimes B}\ \otimes I \quad \begin{array}{c}\mathcal{E}\\ \Delta'', A, B \Vdash C\end{array}}{\Delta, \Delta', \Delta'' \Vdash C}\ \otimes E \quad \stackrel{R}{\Longrightarrow} \quad \begin{array}{c}\mathcal{E}'\\ \Delta, \Delta', \Delta'' \Vdash C\end{array}$$

This is still somewhat informal; a more formal account would label linear hypotheses with distinct variables, introduce proof terms, and carry out the corresponding substitution on the proof terms. We omit this rather standard detail. We also leave out the judgmental label *true* when it is clear from context. Because locally sound rules create no spurious evidence, we can characterize the connectives independently of each other, relying only on judgmental concepts and their introduction and elimination rules. The computational interpretation (β-reduction) of proofs depends on local reduction.

Conversely, we have a notion of local completeness to show that the elimination rules are not too weak: by eliminating a propositional connective we can obtain sufficient evidence to reconstitute the original judgment. We show this by means of a *local expansion* ($\underset{E}{\Longrightarrow}$) that transforms an arbitrary proof of a proposition to one that introduces its main connective.

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \otimes B\end{array} \quad \underset{E}{\Longrightarrow} \quad \frac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \otimes B\end{array} \quad \dfrac{\dfrac{}{A \Vdash A}\ \mathrm{hyp} \quad \dfrac{}{B \Vdash B}\ \mathrm{hyp}}{A, B \Vdash A \otimes B}\ \otimes I}{\Delta \Vdash A \otimes B}\ \otimes E$$

Local expansion provides a source of canonical forms in logical frameworks (η-expansion).

The remaining multiplicative ($\mathbf{1}$, $\multimap$) and additive ($\&$, $\top$, $\oplus$, $\mathbf{0}$) connectives of intuitionistic linear logic have justifications similar to that of $\otimes$ (see Fig. 1). We skip the easy verification of local soundness and completeness of individual rules, but note that we need no further judgmental constructs or principles for this fragment of linear logic.

2.2 Validity

The logic of the previous section permits only linear hypotheses; it is therefore too weak to embed ordinary intuitionistic or classical logic. To allow such embeddings, Girard defined a modal operator ! to allow hereditary translations of ordinary formulas into linear formulas. We start instead with the introduction of a new judgment, "*A is valid*", which we write $A\ \mathit{valid}$. The ! operator is then the internalization of the validity judgment as a truth judgment. Thus, we view validity not as a primitive judgment, but as a *categorical judgment* derived from truth in the absence of linear hypotheses. Similar notions of categorical judgment have been used in a wide variety of logics [13, 33].

If $\cdot \Vdash A\ \mathit{true}$, *then* $A\ \mathit{valid}$.

For the resource interpretation of judgments, $A\ \mathit{valid}$ means that one can achieve $A\ \mathit{true}$ without consuming any resources. Dually, an assumption $A\ \mathit{valid}$ lets one generate as many copies of $A\ \mathit{true}$ as needed, including none at all; structurally, this corresponds to allowing weakening and contraction with hypotheses of the form $A\ \mathit{valid}$. The resulting hypothetical judgment has some *unrestricted hypotheses*, familiar from ordinary logic, and some *linear hypotheses* as described in the preceding section. We write this as a judgment with two zones containing assumptions, following the style introduced by Andreoli [4].

$$\underbrace{A_1\ \mathit{valid}, \ldots, A_j\ \mathit{valid}}_{\Gamma} ; \underbrace{B_1\ \mathit{true}, \ldots, B_k\ \mathit{true}}_{\Delta} \vdash C\ \mathit{true}$$

The use of the semi-colon to separate the two kinds of hypotheses is now standard practice. Our definition of validity makes $\Gamma \vdash A\ \mathit{valid}$ synonymous with $\Gamma ; \cdot \vdash A\ \mathit{true}$; however, because we define connectives via rules about their *truth* rather than *validity*, we avoid using $A\ \mathit{valid}$ as a conclusion of hypothetical judgments. Instead, the categorical view of validity is incorporated in its hypothesis rule.

$$\frac{}{\Gamma, A\ \mathit{valid} ; \cdot \vdash A\ \mathit{true}}\ \mathrm{hyp!}$$

We obtain a second substitution principle to account for unrestricted hypotheses: if we have a proof that $A\ \mathit{valid}$ we can substitute it for assumptions of $A\ \mathit{valid}$. Note that while Δ and Δ' are joined, Γ remains the same, expressing the fact that unrestricted hypotheses may be used multiple times.

Principle 2 (Substitution).

1. *If* $\Gamma ; \Delta \vdash A\ \mathit{true}$ *and* $\Gamma ; \Delta', A\ \mathit{true} \vdash C\ \mathit{true}$, *then* $\Gamma ; \Delta, \Delta' \vdash C\ \mathit{true}$.
2. *If* $\Gamma ; \cdot \vdash A\ \mathit{true}$ *and* $\Gamma, A\ \mathit{valid} ; \Delta \vdash C\ \mathit{true}$, *then* $\Gamma ; \Delta \vdash C\ \mathit{true}$.

The validity judgment is internalized as the modal operator !. The introduction rule makes the modality clear by requiring validity of A in the premiss. For the elimination rule, if one has a proof of $!A$, then one is allowed to use A as an unrestricted hypothesis.

$$\frac{\Gamma ; \cdot \vdash A}{\Gamma ; \cdot \vdash\ !A}\ !I \qquad \frac{\Gamma ; \Delta \vdash\ !A \quad \Gamma, A ; \Delta' \vdash C}{\Gamma ; \Delta, \Delta' \vdash C}\ !E$$

To establish local soundness, we use the new substitution principle to substitute $\mathcal{D}$ for uses of A in $\mathcal{E}$ to obtain $\mathcal{E}'$. For local completeness, we require the new hypothesis rule.

$$\frac{\dfrac{\overset{\mathcal{D}}{\Gamma ; \cdot \vdash A}}{\Gamma ; \cdot \vdash\ !A}\ !I \qquad \overset{\mathcal{E}}{\Gamma, A ; \Delta \vdash C}}{\Gamma ; \Delta \vdash C}\ !E \quad \overset{R}{\Longrightarrow} \quad \overset{\mathcal{E}'}{\Gamma ; \Delta \vdash C}$$

$$
\begin{array}{c}\mathcal{D}\\ \Gamma\,;\Delta \vdash\ !A\end{array}
\quad \underset{E}{\Longrightarrow} \quad
\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\,;\Delta \vdash\ !A\end{array} \quad \dfrac{\dfrac{}{\Gamma, A\,; \cdot \vdash A}\ \text{hyp!}}{\Gamma, A\,; \cdot \vdash\ !A}\ !I}{\Gamma\,;\Delta \vdash\ !A}\ !E
$$

The ! operator introduces a slight redundancy in the logic in the form of the equivalence $\mathbf{1} \dashv\vdash\ !\top$. We leave $\mathbf{1}$ as a primitive in JILL because it has a definition in terms of introduction and elimination rules in the purely multiplicative fragment.

2.3 Possibility

Conclusions so far have been of the form A *true*, which is not sufficient to express negation or contradiction among the hypotheses. In the usual view, contradictory hypotheses describe a condition where an actual proof of the conclusion is unnecessary; quite clearly, such a view violates linearity of the conclusion. An alternate approach would define $\neg A$ as $A \multimap \mathbf{0}$, like in Girard's translation from intuitionistic logic to classical linear logic, but then we give up all pretense of linearity, because A *true* and $\neg A$ *true* prove anything, independently of any other linear hypotheses that may remain.

To develop the notion of *linear contradiction*, we introduce a new judgment of *possibility*. Intuitively, a proof of A *poss either* provides a proof of A *true*, *or* exhibits a linear contradiction among the hypotheses. We allow conclusions of the form A *poss* in hypothetical judgments but eliminate it from consideration among the hypotheses. We characterize A *poss* via its substitution principle.

Principle 3 (Substitution for Possibility).
If $\Gamma\,;\Delta \vdash A$ poss and $\Gamma\,; A$ true $\vdash C$ poss, then $\Gamma\,;\Delta \vdash C$ poss.

We justify this principle as follows. Assume A *poss*. Then, there may exist a contradiction among the assumptions, in which case also C *poss*. On the other hand, we may actually have A *true*, in which case any judgment C *poss* we can derive from it is evident.

Two tempting generalizations of this substitution principle turn out to conflict with linearity. If for the second assumption we admit $\Gamma\,;\Delta', A$ *true* $\vdash C$ *poss*, with the conclusion $\Gamma\,;\Delta,\Delta' \vdash C$ *poss*, then we weaken incorrectly if in fact we did not know that A *true*, but had contradictory hypotheses. The other incorrect generalization would be to replace C *poss* with C *true*. This is also unsound if the assumptions were contradictory, because we would not have any evidence for C *true*.

Our explanation of possibility requires a new rule to conclude A *poss* if A *true*.

$$
\frac{\Gamma\,;\Delta \vdash A\ \textit{true}}{\Gamma\,;\Delta \vdash A\ \textit{poss}}\ \text{poss}
$$

The situation here is dual to the validity judgment; there we added a new hypothesis rule but no explicit rule to conclude A *valid*, because we consider validity only as an assumption. With possibility, we add an explicit rule to conclude A *poss* but no new hypothesis rule, because we consider possibility only as a conclusion.

The previous substitution principles for truth and validity (2) require an update for possibility judgments on the right. We consolidate all cases of the substitution principle below, using J schematically to stand for either C *true* or C *poss*.

Principle 4 (Substitution).

1. *If $\Gamma\,;\Delta \vdash A$ poss and $\Gamma\,; A$ true $\vdash C$ poss, then $\Gamma\,;\Delta \vdash C$ poss.*
2. *If $\Gamma\,;\Delta \vdash A$ true and $\Gamma\,;\Delta', A$ true $\vdash J$, then $\Gamma\,;\Delta,\Delta' \vdash J$.*
3. *If $\Gamma\,;\cdot \vdash A$ true and Γ, A valid $;\Delta \vdash J$, then $\Gamma\,;\Delta \vdash J$.*

Similar to validity, we internalize possibility as a modal operator ?.

$$\frac{\Gamma\,;\Delta \vdash A\ \mathit{poss}}{\Gamma\,;\Delta \vdash ?A\ \mathit{true}}\ ?I \qquad \frac{\Gamma\,;\Delta \vdash ?A\ \mathit{true} \quad \Gamma\,;A\ \mathit{true} \vdash C\ \mathit{poss}}{\Gamma\,;\Delta \vdash C\ \mathit{poss}}\ ?E$$

Local reduction for ? demonstrates the new case of the substitution principle (4.1) applied to $\mathcal{D}$ and $\mathcal{E}$ to obtain $\mathcal{E}'$, while local expansion requires a use of the poss rule.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\,;\Delta \vdash A\ \mathit{poss}\end{array}}{\Gamma\,;\Delta \vdash ?A\ \mathit{true}}\ ?I \qquad \begin{array}{c}\mathcal{E}\\ \Gamma\,;A\ \mathit{true} \vdash C\ \mathit{poss}\end{array}}{\Gamma\,;\Delta \vdash C\ \mathit{poss}}\ ?E \quad \underset{R}{\Longrightarrow} \quad \begin{array}{c}\mathcal{E}'\\ \Gamma\,;\Delta \vdash C\ \mathit{poss}\end{array}$$

$$\begin{array}{c}\mathcal{D}\\ \Gamma\,;\Delta \vdash ?A\ \mathit{true}\end{array} \quad \underset{E}{\Longrightarrow} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\,;\Delta \vdash ?A\ \mathit{true}\end{array} \qquad \dfrac{\dfrac{}{\Gamma\,;A\ \mathit{true} \vdash A\ \mathit{true}}\ \mathrm{hyp}}{\Gamma\,;A\ \mathit{true} \vdash A\ \mathit{poss}}\ \mathrm{poss}}{\Gamma\,;\Delta \vdash A\ \mathit{poss}}\ ?E}{\Gamma\,;\Delta \vdash ?A\ \mathit{true}}\ ?I$$

The system so far with the primitive connectives $\otimes$, $\mathbf{1}$, &, $\top$, $\oplus$, $\mathbf{0}$, !, and ? we call JILL for *judgmental intuitionistic linear logic*. Fig. 1 lists the complete set of rules for these connectives. As a minor complication, certain rules hold regardless of the form of the conclusion – C *true* or C *poss*. We write this using the judgmental schema J; for example, the rule for $\otimes$ elimination has the schematic form

$$\frac{\Gamma\,;\Delta \vdash A \otimes B \quad \Gamma\,;\Delta', A, B \vdash J}{\Gamma\,;\Delta, \Delta' \vdash J}\ \otimes E$$

where we intend J as either C *true* or C *poss*. The substitution principle already uses such a schematic presentation, so all local reductions remain correct. As usual, we omit judgmental labels when apparent.

2.4 Negation and contradiction

Conceptually, in order to prove $\neg A$ *true*, we would like to assume A *true* and derive a contradiction. As remarked before, we want this contradiction to consume resources linearly, in order to distinguish it from the (non-linear) negation of A, definable as $A \multimap \mathbf{0}$. In other words, this contradiction should correspond to multiplicative falsehood, not additive falsehood ($\mathbf{0}$). This suggests a conservative extension of JILL with the following additional judgment form: $\Gamma\,;\Delta \vdash \cdot$, with the meaning that the hypotheses are (linearly) contradictory. One can now give rules for negation and contradiction directly without any need for the possibility judgment; a similar approach was taken by Troelstra for the system ILZ, which is a sequent-style presentation of an intuitionistic fragment of classical linear logic [40]. Instead of taking this path, we relate this new hypothetical judgment form to the possibility judgment in JILL. Thus we obtain a *right-weakening* rule:

$$\frac{\Gamma\,;\Delta \vdash \cdot}{\Gamma\,;\Delta \vdash C\ \mathit{poss}}\ \mathrm{poss}'$$

The pair of rules poss and poss′ completely characterize our interpretation of C *poss* as either C *true* or a condition of contradictory hypotheses. The right-hand side J in various elimination rules (Fig. 1) and substitution principles (4) must extend to allow for this new judgment. In particular, the case for possibility (4.1) must also allow for "$\cdot$" in addition to C *poss*.

Judgmental Rules

$$\frac{}{\Gamma\,;A \vdash A}\ \text{hyp} \qquad \frac{}{\Gamma, A\,;\cdot \vdash A}\ \text{hyp!} \qquad \frac{\Gamma\,;\Delta \vdash A\ \mathit{true}}{\Gamma\,;\Delta \vdash A\ \mathit{poss}}\ \text{poss}$$

Multiplicative Connectives

$$\frac{\Gamma\,;\Delta \vdash A \quad \Gamma\,;\Delta' \vdash B}{\Gamma\,;\Delta, \Delta' \vdash A \otimes B}\ \otimes I \qquad \frac{\Gamma\,;\Delta \vdash A \otimes B \quad \Gamma\,;\Delta', A, B \vdash J}{\Gamma\,;\Delta, \Delta' \vdash J}\ \otimes E$$

$$\frac{}{\Gamma\,;\cdot \vdash \mathbf{1}}\ \mathbf{1}I \qquad \frac{\Gamma\,;\Delta \vdash \mathbf{1} \quad \Gamma\,;\Delta' \vdash J}{\Gamma\,;\Delta, \Delta' \vdash J}\ \mathbf{1}E$$

$$\frac{\Gamma\,;\Delta, A \vdash B}{\Gamma\,;\Delta \vdash A \multimap B}\ \multimap I \qquad \frac{\Gamma\,;\Delta \vdash A \multimap B \quad \Gamma\,;\Delta' \vdash A}{\Gamma\,;\Delta, \Delta' \vdash B}\ \multimap E$$

Additive Connectives

$$\frac{\Gamma\,;\Delta \vdash A \quad \Gamma\,;\Delta \vdash B}{\Gamma\,;\Delta \vdash A \,\&\, B}\ \& I \qquad \frac{\Gamma\,;\Delta \vdash A \,\&\, B}{\Gamma\,;\Delta \vdash A}\ \& E_1 \qquad \frac{\Gamma\,;\Delta \vdash A \,\&\, B}{\Gamma\,;\Delta \vdash B}\ \& E_2$$

$$\frac{}{\Gamma\,;\Delta \vdash \top}\ \top I \qquad \frac{\Gamma\,;\Delta \vdash \mathbf{0}}{\Gamma\,;\Delta, \Delta' \vdash J}\ \mathbf{0}E$$

$$\frac{\Gamma\,;\Delta \vdash A}{\Gamma\,;\Delta \vdash A \oplus B}\ \oplus I_1 \qquad \frac{\Gamma\,;\Delta \vdash B}{\Gamma\,;\Delta \vdash A \oplus B}\ \oplus I_2$$

$$\frac{\Gamma\,;\Delta \vdash A \oplus B \quad \Gamma\,;\Delta', A \vdash J \quad \Gamma\,;\Delta', B \vdash J}{\Gamma\,;\Delta, \Delta' \vdash J}\ \oplus E$$

Exponentials

$$\frac{\Gamma\,;\cdot \vdash A}{\Gamma\,;\cdot \vdash\, !A}\ !I \qquad \frac{\Gamma\,;\Delta \vdash\, !A \quad \Gamma, A\,;\Delta' \vdash J}{\Gamma\,;\Delta, \Delta' \vdash J}\ !E$$

$$\frac{\Gamma\,;\Delta \vdash A\ \mathit{poss}}{\Gamma\,;\Delta \vdash\, ?A}\ ?I \qquad \frac{\Gamma\,;\Delta \vdash\, ?A \quad \Gamma\,;A \vdash C\ \mathit{poss}}{\Gamma\,;\Delta \vdash C\ \mathit{poss}}\ ?E$$

Figure 1: Natural deduction for JILL

Multiplicative contradiction $\bot$ internalizes this new hypothetical judgment, having obvious introduction and elimination rules.

$$\frac{\Gamma\,;\Delta \vdash \cdot}{\Gamma\,;\Delta \vdash \bot\ \mathit{true}}\ \bot I \qquad \frac{\Gamma\,;\Delta \vdash \bot\ \mathit{true}}{\Gamma\,;\Delta \vdash \cdot}\ \bot E$$

Multiplicative negation $\neg$ is also straightforward, having a multiplicative elimination rule.

$$\frac{\Gamma\,;\Delta, A \vdash \cdot}{\Gamma\,;\Delta \vdash \neg A\ \mathit{true}}\ \neg I \qquad \frac{\Gamma\,;\Delta \vdash A\ \mathit{true} \quad \Gamma\,;\Delta' \vdash \neg A\ \mathit{true}}{\Gamma\,;\Delta, \Delta' \vdash \cdot}\ \neg E$$

Note that it is possible to view poss′ as an admissible structural rule of weakening on the right by allowing the conclusion of $\bot E$ and $\neg E$ to be either "$\cdot$" or $C\ \mathit{poss}$. We take this approach in the next section.

Local soundness and completeness of the rules for $\bot$ and $\neg$ are easily verified, so we omit them here. Instead, we note that in JILL, we can define $\bot$ and $\neg$ notationally (as propositions).

$$\bot \stackrel{\text{def}}{=} ?\mathbf{0} \qquad \neg A \stackrel{\text{def}}{=} A \multimap \bot$$

Thus, we have no pressing need to add empty conclusions to JILL in order to define linear contradiction and negation. Of course, in the absence of additive contradiction **0** this definition is impossible. The extension with empty conclusions does not require additive connectives, so negation belongs to the multiplicative-exponential fragment of JILL.

As an example, we present the following proof of $A \otimes \neg A \multimap ?C$ for arbitrary C.

$$
\dfrac{
  \dfrac{}{\cdot\,;A \otimes \neg A \vdash A \otimes \neg A}\;\text{hyp}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\cdot\,;\neg A \vdash \neg A}\;\text{hyp}
        \quad
        \dfrac{}{\cdot\,;A \vdash A}\;\text{hyp}
      }{\cdot\,;A, \neg A \vdash ?\mathbf{0}}\;{\multimap}E
      \quad
      \dfrac{
        \dfrac{}{\cdot\,;\mathbf{0} \vdash \mathbf{0}}\;\text{hyp}
      }{\cdot\,;\mathbf{0} \vdash C\ \ \mathit{poss}}\;\mathbf{0}E
    }{\cdot\,;A, \neg A \vdash C\ \ \mathit{poss}}\;?E
  }{\cdot\,;A, \neg A \vdash ?C}\;?I
}{
  \dfrac{\cdot\,;A \otimes \neg A \vdash ?C}{\cdot\,;\cdot \vdash A \otimes \neg A \multimap ?C}\;{\multimap}I
}\;{\otimes}E
$$

If in this derivation we try to replace $?C$ by C, then the instance of $?E$ becomes inapplicable, since this rule requires possibility on the right. To show formally that indeed $A \otimes \neg A \multimap C$ cannot hold for arbitrary C (*i.e.*, $\neg$ behaves multiplicatively), we rely on the sequent calculus for JILL in Sec. 3.

2.5 A modal disjunction

Because natural deduction *a priori* admits only a single conclusion, a purely multiplicative disjunction seems conceptually difficult. Generalizing the right-hand side to admit more than one *true* proposition would violate either linearity or the intuitionistic interpretation. Yet, multiple conclusions do not necessarily conflict with natural deduction (see, for example, [30]), even for intuitionistic [36] and linear logics [23]. Indeed, we can readily incorporate such an approach in our judgmental framework by introducing a new judgment form, $C_1\ \ \mathit{poss} \mid \cdots \mid C_k\ \ \mathit{poss}$, on the right-hand side of a hypothetical judgment, with the meaning that *either* the hypotheses are contradictory *or* one of the C_i is true. Thus we obtain the following rule for possibility (replacing our previous poss rule)

$$
\dfrac{\Gamma\,;\Delta \vdash C\ \ \mathit{true}}{\Gamma\,;\Delta \vdash C\ \ \mathit{poss} \mid \Sigma}\;\text{poss}
$$

where Σ stands for some alternation $C_1\ \ \mathit{poss} \mid \cdots \mid C_k\ \ \mathit{poss}$, with free exchange assumed for "$\mid$". Since introduction rules define truth, this rule forces a commitment to the truth of a particular proposition, which retains both the intuitionistic and linear character of the logic. Structurally, Σ behaves as the dual of Γ on the right-hand side of hypothetical judgments, with weakening and contraction as admissible structural rules. The multiplicative contradiction sketched in the previous section becomes a special case with *no* possible conclusions, *i.e.*, an empty Σ. We obtain the following generalized structural and substitution principles, where J stands for either $C\ \ \mathit{true}$ or Σ.

Principle 5 (Substitution).

1. *If* $\Gamma\,;\Delta \vdash A\ \ \mathit{poss} \mid \Sigma$ *and* $\Gamma\,;A\ \ \mathit{true} \vdash \Sigma$ *then* $\Gamma\,;\Delta \vdash \Sigma$.
2. *If* $\Gamma\,;\Delta \vdash A\ \ \mathit{true}$ *and* $\Gamma\,;\Delta', A\ \ \mathit{true} \vdash J$ *then* $\Gamma\,;\Delta, \Delta' \vdash J$.
3. *If* $\Gamma\,;\cdot \vdash A\ \ \mathit{true}$ *and* $\Gamma, A\ \ \mathit{valid}\,;\Delta \vdash J$, *then* $\Gamma\,;\Delta \vdash J$.

Armed with this new hypothetical judgment form, we can define a multiplicative and modal disjunction $\boxplus$ with the following rules:

$$
\dfrac{\Gamma\,;\Delta \vdash A\ \ \mathit{poss} \mid B\ \ \mathit{poss}}{\Gamma\,;\Delta \vdash A \boxplus B\ \ \mathit{true}}\;{\boxplus}I
\qquad
\dfrac{\Gamma\,;\Delta \vdash A \boxplus B\ \ \mathit{true} \quad \Gamma\,;A \vdash \Sigma \quad \Gamma\,;B \vdash \Sigma}{\Gamma\,;\Delta \vdash \Sigma}\;{\boxplus}E
$$

By the nature of the poss rule and the substitution principles, this disjunction has both linear and modal aspects. The details of this formuation of a multiple-conclusion JILL is in the appendix.

Like contradiction and negation in the previous section, we can also define $\boxplus$ notationally in JILL as a combination of additive disjunction and the ?-modality.

$$A \boxplus B \quad \stackrel{\text{def}}{=} \quad ?(A \oplus B)$$

This definition allows us to retain our presentation of JILL (Fig. 1) with a single conclusion. On the other hand, this definition requires the additive disjunction $\oplus$, so the definitional view does not hold for the multiplicative-exponential fragment of JILL. In either definition, judgmental or notational, $\boxplus$ does not correspond exactly to the multiplicative disjunction $⅋$ from classical linear logic (CLL) or FILL because of its modal nature. For example, $\perp$ does not function as the unit of $\boxplus$, but we instead have the equivalence $A \boxplus \perp \dashv\vdash ?A$. Associativity and commutativity do hold, however.

2.6 Other connectives and properties

To summarize the preceding two sections, $\perp$ and $\boxplus$ have two different presentations. One requires a generalization of the judgment forms and presents them via their introduction and elimination rules. The other uses ?, $\oplus$, and **0** to define them notationally. The fact that both explanations are viable and coincide confirms their status as logical connectives, not just abbreviations.

Like with intuitionistic logic, we can extend JILL with other connectives, either via introduction and elimination rules or directly via notational definitions. Two new forms of implication corresponding to the two modals ! and ? appear particularly useful.

	proposition	**definition**
unrestricted implication	$A \supset B$	$!A \multimap B$
partial implication	$A \rightharpoonup B$	$A \multimap ?B$

Under the Curry-Howard isomorphism, $A \supset B$ corresponds to the type of function that uses its argument arbitrarily often, possibly never. Similarly, $A \rightharpoonup B$ corresponds to a *linear partial* function from type A to B. Even types such as $!A \multimap ?B$ can be given a sensible interpretation, in this case simply the partial functions from A to B. We expect the ? modality and various partial function types to be particularly useful for programming languages with recursion (and perhaps also effectful computations) in a setting where one does not wish to sacrifice linearity entirely.

We close this section with a statement of the structural properties and the validity of the substitution principle.

Theorem 1 (Structural Properties). *JILL satisfies the substitution principles 4. Additionally, the following structural properties hold.*

1. *If* $\Gamma\,;\Delta \vdash C\ \ true$ *then* $\Gamma, A\ \ valid\,;\Delta \vdash C\ \ true.$ *(weakening)*
2. *If* $\Gamma, A\ \ valid, A\ \ valid\,;\Delta \vdash C\ \ true$ *then* $\Gamma, A\ \ valid\,;\Delta \vdash C\ \ true.$ *(contraction)*

Proof. By straightforward structural inductions. The substitution principle for possibility

If $\Gamma\,;\Delta \vdash A\ \ poss$ and $\Gamma\,;A\ \ true \vdash C\ \ poss$, then $\Gamma\,;\Delta \vdash C\ \ poss$.

requires a somewhat unusual proof by induction over the structure of both given derivations, not just the second. This is not unexpected, however, since this proof is analogous to a similar proof for the judgmental system for intuitionistic modal logic [33]. □

3 Sequent calculus for JILL

Critical to the understanding of logical connectives is that the meaning of a proposition depends only on its constituents. Martin-Löf states [27, Page 27] that "*the meaning of a proposition is determined by [...] what counts as a verification of it*", but he does not elaborate on the notion of a verification. It seems clear that as a minimal condition, verifications must refer only to the propositions constituting the judgment they establish. In other words, they must obey the subformula property. We argue for a stronger condition that comes directly from the justification of rules, where we do not refer to any extraneous features of any particular proof.

> *Every logical inference in a verification must proceed purely by decomposition of one logical connective.*

For the natural deduction view, introduction rules should only decompose the goal one is trying to achieve, while elimination rules should only decompose the assumptions one has. Since introduction rules have the subformula property when read bottom-up, and elimination rules have the subformula property when read top-down, any proof in this style will therefore satisfy this condition. The notion of a verification can thus be formalized directly on natural deductions (see, for example, [39]). We take a different approach in this paper and formalize verifications as *cut-free proofs in the sequent calculus*. Not only is it immediately evident that the sequent calculus satisfies our condition, but it is easier to prove the correctness of the interpretations of classical linear logic in Sec. 4, which is usually presented in the form of sequents.

The fundamental transformation giving rise to the sequent calculus is a bifurcation of the judgment $A\ \ true$ into judgments for resources and goals, $A\ \ res$ and $A\ \ goal$. We then consider linear hypothetical judgments of the form

$$B_1\ \ res, \ldots, B_k\ \ res \Vdash C\ \ goal.$$

We never consider $C\ \ goal$ as a hypothesis or $B\ \ res$ as a conclusion. Therefore, we do not have a hypothesis rule with the same judgment on both sides of $\Vdash$, like for natural deduction. Rather, if we have the resource A, we can achieve goal A; this we state as a rule relating the judgments $A\ \ res$ and $A\ \ goal$.

$$\frac{}{A\ \ res \Vdash A\ \ goal}\ \text{init}$$

Because we do not allow resources on the right and goals on the left, we cannot write its dual as $A\ \ goal \Vdash A\ \ res$. Instead, we obtain a form of cut as a proper dual of the init rule.

Principle 6 (Cut). *If $\Delta \Vdash A\ \ goal$ and $\Delta', A\ \ res \Vdash C\ \ goal$, then $\Delta, \Delta' \Vdash C\ \ goal$.*

In words, if we have achieved a goal A, then we may justifiably use A as a resource. Because we distinguish resources and goals, cut does not correspond exactly to the substitution principles of natural deduction. But, similar to the substitution principles, cut must remain an admissible rule in order to preserve our view of verification; if not, a proof might refer to an arbitrary cut-formula A that does not occur in the conclusion. Similarly, if we did not distinguish two judgments, the interpretation of hypothetical judgments would force the collapse back to natural deduction.

To capture all of JILL as a sequent calculus, we also need to account for validity and possibility. Fortunately, we already restrict their occurrence in hypothetical judgments – to the left for validity and to the right for possibility. These judgments therefore do not require splits. Thus, we obtain the following general hypothetical judgment forms, called *sequents* following Gentzen's terminology.

$$A_1\ \ valid, \ldots, A_j\ \ valid\ ;\ B_1\ \ res, \ldots, B_k\ \ res \Longrightarrow C\ \ goal$$

$$A_1 \ \mathit{valid}, \ldots, A_j \ \mathit{valid} \,; B_1 \ \mathit{res}, \ldots, B_k \ \mathit{res} \Longrightarrow C \ \mathit{poss}$$

As before, we write the left-hand side schematically as $\Gamma \,; \Delta$. The division of the left-hand side into zones allows us to leave the judgment labels implicit. We employ a symmetric device on the right-hand side, representing $C \ \mathit{goal}$ by $C \,; \cdot$ and $C \ \mathit{poss}$ by $\cdot \,; C$. For left rules where the actual form of the right-hand side often does not matter, we write it schematically as γ. We use $\Longrightarrow$ instead of $\Vdash$ or $\vdash$ for the hypothetical judgment to visually distinguish the sequent calculus from natural deduction.

We now systematically construct the rules defining the judgments of the sequent calculus. The introduction rules from natural deduction turn into *right rules* that operate only on goals and retain their bottom-up interpretation. For the elimination rules, we reverse the direction and have them operate only on resources, thus turning them into *left rules*, also read bottom-up. Rules in the sequent calculus therefore have a uniform bottom-up interpretation, unlike the rules for natural deduction. With the inclusion of *valid* and *poss* judgments, the init rule has the following most general form.

$$\frac{}{\Gamma \,; A \Longrightarrow A \,; \cdot} \ \text{init}$$

We allow copying valid hypotheses in Γ into the linear context Δ (reading bottom-up) by means of a copy rule. This rule corresponds to hyp! of natural deduction.

$$\frac{\Gamma, A \,; \Delta, A \Longrightarrow \gamma}{\Gamma, A \,; \Delta \Longrightarrow \gamma} \ \text{copy}$$

Finally, we include the counterpart of the poss rule, which in the sequent calculus promotes a possibility goal $\cdot \,; C$ in the conclusion into a true goal $C \,; \cdot$ in the premiss.

$$\frac{\Gamma \,; \Delta \Longrightarrow C \,; \cdot}{\Gamma \,; \Delta \Longrightarrow \cdot \,; C} \ \text{promote}$$

Fig. 2 lists the rules for the various connectives. Structurally, weakening and contraction of the valid context Γ continue to hold.

Theorem 2 (Structural Properties).

1. *If* $\Gamma \,; \Delta \Longrightarrow \gamma$, *then* $\Gamma, A \,; \Delta \Longrightarrow \gamma$. *(weakening)*
2. *If* $\Gamma, A, A \,; \Delta \Longrightarrow \gamma$, *the* $\Gamma, A \,; \Delta \Longrightarrow \gamma$. *(contraction)*

The proof is by structural induction on the given derivations; we omit the easy verification. Cut comes in three forms, dualizing init, copy, and promote, respectively.

Principle 7 (Cut).

1. *If* $\Gamma \,; \cdot \Longrightarrow A \,; \cdot$ *and* $\Gamma, A \,; \Delta' \Longrightarrow \gamma$, *then* $\Gamma \,; \Delta' \Longrightarrow \gamma$.
2. *If* $\Gamma \,; \Delta \Longrightarrow A \,; \cdot$ *and* $\Gamma \,; \Delta', A \Longrightarrow \gamma$, *then* $\Gamma \,; \Delta, \Delta' \Longrightarrow \gamma$.
3. *If* $\Gamma \,; \Delta \Longrightarrow \cdot \,; A$ *and* $\Gamma \,; A \Longrightarrow \cdot \,; C$, *then* $\Gamma \,; \Delta \Longrightarrow \cdot \,; C$.

It is important to note that as a direct result of the separation of valid and linear hypotheses, no further kinds of cut are needed. Without this separation, the number and complexity of cut rules—and the corresponding cut-admissibility proof—become quite large; Howe has a detailed proof for such a single-zoned sequent system for intuitionistic linear logic (without possibility, negation or modal disjunction) [22]. Our proof of cut admissibility is more direct, using lexicographic induction on the relevant derivations.

Judgmental Rules

$$\frac{}{\Gamma\,;A \Longrightarrow A\,;\cdot}\ \text{init} \qquad \frac{\Gamma, A\,;\Delta, A \Longrightarrow \gamma}{\Gamma, A\,;\Delta \Longrightarrow \gamma}\ \text{copy} \qquad \frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot}{\Gamma\,;\Delta \Longrightarrow \cdot\,;A}\ \text{promote}$$

Multiplicative Connectives

$$\frac{\Gamma\,;\Delta, A, B \Longrightarrow \gamma}{\Gamma\,;\Delta, A \otimes B \Longrightarrow \gamma}\ {\otimes}L \qquad \frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot \quad \Gamma\,;\Delta' \Longrightarrow B\,;\cdot}{\Gamma\,;\Delta, \Delta' \Longrightarrow A \otimes B\,;\cdot}\ {\otimes}R$$

$$\frac{\Gamma\,;\Delta \Longrightarrow \gamma}{\Gamma\,;\Delta, \mathbf{1} \Longrightarrow \gamma}\ \mathbf{1}L \qquad \frac{}{\Gamma\,;\cdot \Longrightarrow \mathbf{1}\,;\cdot}\ \mathbf{1}R$$

$$\frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot \quad \Gamma\,;\Delta', B \Longrightarrow \gamma}{\Gamma\,;\Delta, \Delta', A \multimap B \Longrightarrow \gamma}\ {\multimap}L \qquad \frac{\Gamma\,;\Delta, A \Longrightarrow B\,;\cdot}{\Gamma\,;\Delta \Longrightarrow A \multimap B\,;\cdot}\ {\multimap}R$$

Additive Connectives

$$\frac{\Gamma\,;\Delta, A \Longrightarrow \gamma}{\Gamma\,;\Delta, A \,\&\, B \Longrightarrow \gamma}\ \&L_1 \qquad \frac{\Gamma\,;\Delta, B \Longrightarrow \gamma}{\Gamma\,;\Delta, A \,\&\, B \Longrightarrow \gamma}\ \&L_2$$

$$\frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot \quad \Gamma\,;\Delta \Longrightarrow B\,;\cdot}{\Gamma\,;\Delta \Longrightarrow A \,\&\, B\,;\cdot}\ \&R$$

$$\frac{}{\Gamma\,;\Delta \Longrightarrow \top\,;\cdot}\ \top R \qquad \frac{}{\Gamma\,;\Delta, \mathbf{0} \Longrightarrow \gamma}\ \mathbf{0}L$$

$$\frac{\Gamma\,;\Delta, A \Longrightarrow \gamma \quad \Gamma\,;\Delta, B \Longrightarrow \gamma}{\Gamma\,;\Delta, A \oplus B \Longrightarrow \gamma}\ {\oplus}L$$

$$\frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot}{\Gamma\,;\Delta \Longrightarrow A \,\&\, B\,;\cdot}\ {\oplus}R_1 \qquad \frac{\Gamma\,;\Delta \Longrightarrow B\,;\cdot}{\Gamma\,;\Delta \Longrightarrow A \,\&\, B\,;\cdot}\ {\oplus}R_2$$

Exponentials

$$\frac{\Gamma, A\,;\Delta \Longrightarrow \gamma}{\Gamma\,;\Delta, !A \Longrightarrow \gamma}\ !L \qquad \frac{\Gamma\,;\cdot \Longrightarrow A\,;\cdot}{\Gamma\,;\cdot \Longrightarrow !A\,;\cdot}\ !R \qquad \frac{\Gamma\,;A \Longrightarrow \cdot\,;C}{\Gamma\,;?A \Longrightarrow \cdot\,;C}\ ?L \qquad \frac{\Gamma\,;\Delta \Longrightarrow \cdot\,;A}{\Gamma\,;\Delta \Longrightarrow ?A\,;\cdot}\ ?R$$

Figure 2: Sequent calculus for JILL

Theorem 3. *The above principles of cut are admissible rules in JILL.*

Proof. First we name all the derivations (writing $\mathcal{D} :: J$ if $\mathcal{D}$ is a derivation of the judgment J):

$$\begin{array}{llll} \mathcal{D}_1 :: \Gamma\,;\cdot \Longrightarrow A\,;\cdot & \mathcal{E}_1 :: \Gamma, A\,;\Delta' \Longrightarrow \gamma & \mathcal{F}_1 :: \Gamma, \Delta';\Longrightarrow \gamma & \text{(row 1)} \\ \mathcal{D}_2 :: \Gamma\,;\Delta \Longrightarrow A\,;\cdot & \mathcal{E}_2 :: \Gamma\,;\Delta', A \Longrightarrow \gamma & \mathcal{F}_2 :: \Gamma\,;\Delta, \Delta' \Longrightarrow \gamma & \text{(row 2)} \\ \mathcal{D}_3 :: \Gamma\,;\Delta \Longrightarrow \cdot\,;A & \mathcal{E}_3 :: \Gamma\,;A \Longrightarrow \cdot\,;C & \mathcal{F}_3 :: \Gamma\,;\Delta \Longrightarrow \cdot\,;C & \text{(row 3)} \end{array}$$

The computational content of the proof is a way to transform corresponding $\mathcal{D}$ and $\mathcal{E}$ into $\mathcal{F}$. For the purposes of the inductive argument, we may appeal to the induction hypotheses whenever

a. the cut-formula A is strictly smaller;

b. the cut-formula A remains the same, but we select the inductive hypothesis from row 3 for proofs in rows 2 and 1, or from row 2 in proofs of row 1;

c. the cut-formula A and the derivation $\mathcal{E}_i$ remains the same, but the derivation $\mathcal{D}_i$ becomes smaller; or *d.* the cut-formula A and the derivation $\mathcal{D}_i$ remains the same, but the derivation $\mathcal{E}_i$ becomes smaller.

The cases in the inductive proof fall into the following classes, which we will explicitly name and for which we provide a characteristic case.

Initial Cuts. Here, we find an initial sequent in one of the two premisses, so we eliminate these cases directly. For example,

$$\mathcal{D}_2 = \frac{}{\Gamma ; A \Longrightarrow A ; \cdot}\ \text{init} \qquad\qquad \mathcal{E}_2 \text{ arbitrary}$$

$\Delta = A$	this case
$\Gamma ; \Delta', A \Longrightarrow \gamma$	derivation $\mathcal{E}_2$

Principal Cuts. The cut-formula A was just inferred by a right rule in the first premiss and a left rule in the second premiss. In these cases, we appeal to the induction hypotheses, possibly more than once on smaller cut-formulas. For example,

$$\mathcal{D}_2 = \frac{\overset{\mathcal{D}_2'}{\Gamma ; \Delta \Longrightarrow \cdot ; A}}{\Gamma ; \Delta \Longrightarrow ?A ; \cdot}\ ?R \qquad\qquad \mathcal{E}_2 = \frac{\overset{\mathcal{E}_2'}{\Gamma ; A \Longrightarrow \cdot ; C}}{\Gamma ; ?A \Longrightarrow \cdot ; C}\ ?L$$

$\Gamma ; \Delta \Longrightarrow \cdot ; C$	by i.h. (row 3) (case a) on A, $\mathcal{D}_2'$ and $\mathcal{E}_2'$

Copy Cut. We treat the cases for the cut! rule as right commutative cuts (below), except for the copy rule, where we require an appeal to an induction hypothesis on the same cut-formula.

$$\mathcal{D}_1 \text{ arbitrary} \qquad\qquad \mathcal{E}_1 = \frac{\overset{\mathcal{E}_1'}{\Gamma, A ; \Delta', A \Longrightarrow \gamma}}{\Gamma, A ; \Delta' \Longrightarrow \gamma}\ \text{copy}$$

$\Gamma ; \cdot \Longrightarrow A ; \cdot$	derivation $\mathcal{D}_1$
$\Gamma ; \Delta', A \Longrightarrow \gamma$	by i.h. (row 1) (case d) on A, $\mathcal{D}_1$, and $\mathcal{E}_1'$
$\Gamma ; \Delta' \Longrightarrow \gamma$	by i.h. (row 2) (case b) on A, $\mathcal{D}_1$, and above

Promote Cut. We treat the cases for the cut? rule as left commutative cuts (below), except for the promote rule, where we appeal to an inductive hypothesis with the same cut-formula.

$$\mathcal{D}_3 = \frac{\overset{\mathcal{D}_3'}{\Gamma ; \Delta \Longrightarrow A ; \cdot}}{\Gamma ; \Delta \Longrightarrow \cdot ; A}\ \text{promote} \qquad\qquad \mathcal{E}_3 \text{ arbitrary}$$

$\Gamma ; A \Longrightarrow \gamma$	derivation $\mathcal{E}_3$
$\Gamma ; \Delta \Longrightarrow \gamma$	by i.h. (row 2) (case c) on A, $\mathcal{D}_3'$, and $\mathcal{E}_3$.

Left Commutative Cuts. The cut-formula A exists as a side-formula of the last inference rule used in the derivation of the left premiss. In these cases, we appeal to the induction hypotheses with the same cut-formula but smaller left derivation. For example,

$$\mathcal{D}_2 = \dfrac{\begin{matrix}\mathcal{D}_2' \\ \Gamma\,;\Delta, B_1 \Longrightarrow A\,;\cdot\end{matrix}}{\Gamma\,;\Delta, B_1 \mathbin{\&} B_2 \Longrightarrow A\,;\cdot}\ \&L_1 \qquad\qquad \mathcal{E}_2 \text{ arbitrary}$$

$\Gamma\,;\Delta', A \Longrightarrow \gamma$	derivation $\mathcal{E}_2$
$\Gamma\,;\Delta, \Delta', B_1 \Longrightarrow \gamma$	by i.h. (row 2) (case c) on A, $\mathcal{D}_2'$, and $\mathcal{E}_2$
$\Gamma\,;\Delta, \Delta', B_1 \mathbin{\&} B_2 \Longrightarrow \gamma$	by $\&L_1$

Right Commutative Cuts. The cut-formula A exists as a side-formula of the last inference rule used in the derivation of the right premiss. In these cases, we appeal to the induction hypotheses with the same cut-formula but smaller right derivation. For example,

$$\mathcal{D}_2 \text{ arbitrary} \qquad\qquad \mathcal{E}_2 = \dfrac{\begin{matrix}\mathcal{E}_2' \\ \Gamma\,;\Delta', A \Longrightarrow C_1\,;\cdot\end{matrix}}{\Gamma\,;\Delta', A \Longrightarrow C_1 \oplus C_2\,;\cdot}\ \oplus R_1$$

$\Gamma\,;\Delta \Longrightarrow A\,;\cdot$	derivation $\mathcal{D}_2$
$\Gamma\,;\Delta, \Delta' \Longrightarrow C_1\,;\cdot$	by i.h. (row 2) (case d) on A, $\mathcal{D}_2$, and $\mathcal{E}_2'$
$\Gamma\,;\Delta, \Delta' \Longrightarrow C_1 \oplus C_2\,;\cdot$	by $\oplus R_1$

All cases in the induction belong to one of these categories. □

Comparing this proof of cut-admissibility for JILL with other proofs of cut-admissibility or cut-elimination in the literature, it is worth remarking that a nested structural induction suffices. No additional restrictions on the cut rules or induction measures are required. Similar structural proofs for cut-admissibility have been demonstrated for classical linear logic [31], classical and intuitionistic uniform sequent calculi [28, 29] and ordered logic [35].

Sequents have valid interpretations as natural deduction judgments, which we state as a soundness theorem for the sequent calculus.

Theorem 4 (Soundness of $\Longrightarrow$ wrt $\vdash$).

1. *If $\Gamma\,;\Delta \Longrightarrow C\,;\cdot$, then $\Gamma\,;\Delta \vdash C$ true.*
2. *If $\Gamma\,;\Delta \Longrightarrow \cdot\,;C$, then $\Gamma\,;\Delta \vdash C$ poss.*

Proof. Simultaneous induction on the derivations $\mathcal{D}_1 :: \Gamma;\Delta \Longrightarrow C;\cdot$ and $\mathcal{D}_2 :: \Gamma;\Delta \Longrightarrow \cdot;C$. For the right rules, we appeal to the induction hypothesis and apply the corresponding introduction rule. For the left rules, we either directly construct a derivation or appeal to the substitution principle after applying the inductive hypothesis. We show in detail some representative cases:

Case: The last rule in $\mathcal{D}_1$ is init, *i.e.*,

$$\mathcal{D}_1 = \dfrac{}{\Gamma\,;A \Longrightarrow A\,;\cdot}\ \text{init}$$

$\Gamma\,;A \vdash A$ *true*	by hyp

Case: The last rule in $\mathcal{D}_1$ or $\mathcal{D}_2$ is copy, *i.e.*,

$$\mathcal{D}_1 \text{ or } \mathcal{D}_2 = \dfrac{\begin{matrix}\mathcal{D}' \\ \Gamma, A \,;\, \Delta, A \Longrightarrow \gamma\end{matrix}}{\Gamma, A \,;\, \Delta \Longrightarrow \gamma}\ \text{copy}$$

$\Gamma, A \,;\, \Delta, A \vdash \gamma$	by the i.h. on $\mathcal{D}'$
$\Gamma, A \,;\, \cdot \vdash A\ \ \mathit{true}$	by hyp!
$\Gamma, A \,;\, \Delta \vdash \gamma$	by the substitution principle for truth (4.2)

Case: The last rule in $\mathcal{D}_2$ is promote, *i.e.*,

$$\mathcal{D}_2 = \dfrac{\begin{matrix}\mathcal{D}_2' \\ \Gamma \,;\, \Delta \Longrightarrow C \,;\, \cdot\end{matrix}}{\Gamma \,;\, \Delta \Longrightarrow \cdot \,;\, C}\ \text{promote}$$

$\Gamma \,;\, \Delta \vdash C\ \ \mathit{true}$	by the i.h. on $\mathcal{D}_2'$
$\Gamma \,;\, \Delta \vdash C\ \ \mathit{poss}$	by poss

Case: The last rule in $\mathcal{D}_2$ is $?L$, *i.e.*,

$$\mathcal{D}_2 = \dfrac{\begin{matrix}\mathcal{D}_2' \\ \Gamma \,;\, A \Longrightarrow \cdot \,;\, C\end{matrix}}{\Gamma \,;\, ?A \Longrightarrow \cdot \,;\, C}\ ?L$$

$\Gamma \,;\, A \vdash C\ \ \mathit{poss}$	by the i.h. on $\mathcal{D}_2'$
$\Gamma \,;\, ?A \vdash ?A\ \ \mathit{true}$	by hyp
$\Gamma \,;\, ?A \vdash C\ \ \mathit{poss}$	by $?E$

Case: The last rule in $\mathcal{D}_1$ or $\mathcal{D}_2$ is $\multimap L$, *i.e.*,

$$\mathcal{D}_1 \text{ or } \mathcal{D}_2 = \dfrac{\begin{matrix}\mathcal{D}' \\ \Gamma \,;\, \Delta \Longrightarrow A \,;\, \cdot\end{matrix} \quad \begin{matrix}\mathcal{D}'' \\ \Gamma \,;\, \Delta', B \Longrightarrow \gamma\end{matrix}}{\Gamma \,;\, \Delta, \Delta', A \multimap B \Longrightarrow \gamma}\ \multimap L$$

Define $J = C\ \ \mathit{true}$ if γ is $C \,;\, \cdot$ and $J = C\ \ \mathit{poss}$ if γ is $\cdot \,;\, C$. Then,

$\Gamma \,;\, \Delta \vdash A\ \ \mathit{true}$	by the i.h. on $\mathcal{D}'$
$\Gamma \,;\, A \multimap B \vdash A \multimap B\ \ \mathit{true}$	by hyp
$\Gamma \,;\, \Delta, A \multimap B \vdash B\ \ \mathit{true}$	by $\multimap E$
$\Gamma \,;\, \Delta', B \vdash J$	by the i.h. on $\mathcal{D}''$
$\Gamma \,;\, \Delta, \Delta', A \multimap B \vdash J$	by the substitution principle for truth (4.2)

Case: The last rule in $\mathcal{D}_2$ is $?R$, *i.e.*,

$$\mathcal{D}_2 = \dfrac{\begin{matrix}\mathcal{D}_2' \\ \Gamma \,;\, \Delta \Longrightarrow \cdot \,;\, A\end{matrix}}{\Gamma \,;\, \Delta \Longrightarrow ?A \,;\, \cdot}\ ?R$$

$\Gamma \,;\, \Delta \vdash A\ \ \mathit{poss}$	by the i.h. on $\mathcal{D}_2'$
$\Gamma \,;\, \Delta \vdash ?A\ \ \mathit{true}$	by $?I$

All remaining cases have one of the above patterns. □

From the judgmental point of view, this defines a *global completeness* property for natural deduction: every judgment that has a verification can indeed be proven. Conversely, the cut-free sequent calculus is complete with respect to natural deduction. For this direction, we need the admissibility of cut.

Theorem 5 (Completeness of $\Longrightarrow$ wrt $\vdash$).

1. *If* $\Gamma\,;\Delta \vdash A$ *true then* $\Gamma\,;\Delta \Longrightarrow A\,;\cdot$.
2. *If* $\Gamma\,;\Delta \vdash A$ *poss, then* $\Gamma\,;\Delta \Longrightarrow \cdot\,;A$.

Proof. Simultaneous induction on the structure of the derivations $\mathcal{D}_1 :: \Gamma\,;\Delta \vdash A$ *true* and $\mathcal{D}_2 :: \Gamma\,;\Delta \vdash A$ *poss*. The cases for the introduction rules (resp. poss) are mapped directly to the corresponding right rules (resp. promote). For the elimination rules we appeal to cut-admissibility for truth (Thm. 3) to cut out the connective being eliminated. The following are some representative cases.

Case: The last rule in $\mathcal{D}_1$ is hyp!, *i.e.,*

$$\mathcal{D}_1 = \frac{}{\Gamma, A\,;\cdot \vdash A}\ \text{hyp!}$$

$\Gamma, A\,;A \Longrightarrow A\,;\cdot$	by init
$\Gamma, A\,;\cdot \Longrightarrow A\,;\cdot$	by copy

Case: The last rule in $\mathcal{D}_1$ is hyp, *i.e.,*

$$\mathcal{D}_1 = \frac{}{\Gamma\,;A \vdash A}\ \text{hyp}$$

$\Gamma\,;A \Longrightarrow A\,;\cdot$	by init

Case: The last rule in $\mathcal{D}_2$ is poss, *i.e.,*

$$\mathcal{D}_2 = \frac{\begin{array}{c}\mathcal{D}_2'\\ \Gamma\,;\Delta \vdash A\end{array}}{\Gamma\,;\Delta \vdash A\ \mathit{poss}}\ \text{poss}$$

$\Gamma\,;\Delta \Longrightarrow A\,;\cdot$	by i.h. on $\mathcal{D}_2'$
$\Gamma\,;\Delta \Longrightarrow \cdot\,;A$	by promote

Case: The last rule in $\mathcal{D}_1$ is a multiplicative elimination rule, say $\multimap E$.

$$\mathcal{D}_1 = \frac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma\,;\Delta \vdash A \multimap B\end{array} \quad \begin{array}{c}\mathcal{D}_1''\\ \Gamma\,;\Delta' \vdash A\end{array}}{\Gamma\,;\Delta, \Delta' \vdash B}\ \multimap E$$

$\Gamma\,;A \Longrightarrow A\,;\cdot$	by init
$\Gamma\,;B \Longrightarrow B\,;\cdot$	by init
$\Gamma\,;A, A \multimap B \Longrightarrow B\,;\cdot$	by $\multimap L$
$\Gamma\,;\Delta \Longrightarrow A \multimap B\,;\cdot$	by i.h. on $\mathcal{D}_1'$
$\Gamma\,;A, \Delta \Longrightarrow B\,;\cdot$	by cut (7.2) with cut-formula $A \multimap B$
$\Gamma\,;\Delta' \Longrightarrow A\,;\cdot$	by i.h. on $\mathcal{D}_1''$
$\Gamma\,;\Delta', \Delta \Longrightarrow B\,;\cdot$	by cut (7.2) with cut-formula A

Case: The last rule in $\mathcal{D}_1$ is an additive elimination rule, say $\& E_1$.

$$\mathcal{D}_1 = \frac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma\,;\Delta \vdash A \,\&\, B\end{array}}{\Gamma\,;\Delta \vdash A}\ \& E_1$$

$$\begin{array}{ll}
\Gamma\,;A \Longrightarrow A\,;\cdot & \text{by init}\\
\Gamma\,;A \mathbin{\&} B \Longrightarrow A\,;\cdot & \text{by } \&L_1\\
\Gamma\,;\Delta \Longrightarrow A \mathbin{\&} B\,;\cdot & \text{by i.h. on } \mathcal{D}_1'\\
\Gamma\,;\Delta \Longrightarrow A\,;\cdot & \text{by cut (7.2) with cut-formula } A \mathbin{\&} B
\end{array}$$

Case: The last rule in $\mathcal{D}_1$ is $!E$, *i.e.*,

$$\mathcal{D}_1 = \dfrac{\begin{array}{cc}\mathcal{D}_1' & \mathcal{D}_1''\\ \Gamma\,;\Delta \vdash {!A} & \Gamma, A\,;\Delta' \vdash J\end{array}}{\Gamma\,;\Delta,\Delta' \vdash J}\ !E$$

Define $\gamma = C\,;\cdot$ if J is C *true* and $\gamma = \cdot; C$ if J is C *poss*. Then,

$$\begin{array}{ll}
\Gamma, A\,;\Delta' \Longrightarrow \gamma & \text{by i.h. on } \mathcal{D}_1''\\
\Gamma\,;\Delta', {!A} \Longrightarrow \gamma & \text{by } !L\\
\Gamma\,;\Delta \Longrightarrow {!A}\,;\cdot & \text{by i.h. on } \mathcal{D}_1'\\
\Gamma\,;\Delta,\Delta' \Longrightarrow \gamma & \text{by cut (7.2) with cut-formula } !A
\end{array}$$

Case: The last rule in $\mathcal{D}_2$ is $?E$, *i.e.*,

$$\mathcal{D}_2 = \dfrac{\begin{array}{cc}\mathcal{D}_2' & \mathcal{D}_2''\\ \Gamma\,;\Delta \vdash {?A} & \Gamma\,;A \vdash C\ \ poss\end{array}}{\Gamma\,;\Delta \vdash C\ \ poss}\ ?E$$

$$\begin{array}{ll}
\Gamma\,;A \Longrightarrow \cdot\,;C & \text{by i.h. on } \mathcal{D}_2''\\
\Gamma\,;{?A} \Longrightarrow \cdot\,;C & \text{by } ?L\\
\Gamma\,;\Delta \Longrightarrow {?A}\,;\cdot & \text{by i.h. on } \mathcal{D}_2'\\
\Gamma\,;\Delta \Longrightarrow \cdot\,;C & \text{by cut (7.2) with cut-formula } ?A
\end{array}$$

All remaining cases have one of the above patterns. □

This completeness theorem for cut-free sequent derivations proves a *global soundness* theorem for natural deduction: every judgment that has a natural deduction has a cut-free sequent derivation, *i.e.*, it has a verification.

It is also possible to split the judgment A *true* into A *intro* and A *elim*, roughly meaning that it has been established by an introduction or elimination rule, respectively. We can then define a normal form for natural deductions in which one can go from eliminations to introductions but not vice versa, which guarantees the subformula property. By adding commuting reductions to the local reductions, one can reduce every natural deduction to a normal form that can serve as a verification. On the whole we find this approach less perspicuous and harder to justify; thus our choice of the sequent calculus.

The cut-free sequent calculus is an easy source of theorems about JILL. For example, if we want to verify that $A \otimes \neg A \multimap C$ *true* cannot be proven for parameters A and C, we simply explore all possibilities for a derivation of $\cdot\,;\cdot \Longrightarrow A \otimes \neg A \multimap C\,;\cdot$, all of which fail after only a few steps. The sequent calculus is also a point of departure for designing theorem proving procedures for intuitionistic linear logic. We note below the inversion properties that are useful for proofs in the next section. Invertible rules of JILL are rules for which the premisses are derivable if the conclusion is derivable; of these, *weakly invertible* rules are invertible only when the conclusion has no linear hypotheses, with the possible exception of the principal formula. For any rule R, we use R^{-1} to denote its inverse; for binary R the two inverted rules are written R_1^{-1} and R_2^{-1}.

Lemma 6. *The following are the invertible, weakly invertible and non-invertible logical rules of JILL.*

invertible	*weakly invertible*	*non-invertible*
$\otimes L$, $\mathbf{1}L$	$\mathbf{1}R$	$\otimes R$
$\multimap R$		$\multimap L$
$\& R$, $\top R$		$\& L_1$, $\& L_2$
$\oplus L$, $\mathbf{0}L$		$\oplus R_1$, $\oplus R_2$
$!L$, $?R$	$!R$, $?L$	

Proof. For invertible and weakly invertible rules we give a proof of the admissible inverted rule using the admissibility of cut (Thm. 3). For the non-invertible rules we give a counter-example which has the identical principal connective on both sides of the arrow. The following cases are representative.

Case: $\multimap R$ is invertible, *i.e.*, the following rule is derivable or admissible:

$$\frac{\Gamma\,;\Delta \Longrightarrow A \multimap B\,;\cdot}{\Gamma\,;\Delta, A \Longrightarrow B\,;\cdot}\ \multimap R^{-1}$$

$\Gamma\,;A \Longrightarrow A\,;\cdot$ and $\Gamma\,;B \Longrightarrow B\,;\cdot$	by init
$\Gamma\,;A, A \multimap B \Longrightarrow B\,;\cdot$	by $\multimap L$
$\Gamma\,;\Delta \Longrightarrow A \multimap B\,;\cdot$	by premiss
$\Gamma\,;\Delta, A \Longrightarrow B\,;\cdot$	by cut (Thm. 3) with cut-formula $A \multimap B$

Case: $\multimap L$ is not invertible in general. Consider $\cdot\,;A \multimap B \Longrightarrow A \multimap B\,;\cdot$, which is initial. However, the premisses of $\multimap L$ for this sequent, *viz.* $\cdot\,;\cdot \Longrightarrow A\,;\cdot$ and $\cdot\,;B \Longrightarrow A \multimap B\,;\cdot$, are not derivable in general. □

4 Interpreting classical linear logic

It is well known that intuitionistic logic is more expressive than classical logic because it makes finer distinctions and therefore has a richer set of connectives. This observation is usually formalized via a translation from classical logic to intuitionistic logic that preserves provability. A related argument has been made by Girard [16] regarding classical linear logic: it is more expressive than both intuitionistic and classical logic since we can easily interpret both of these. In this section we show that intuitionistic linear logic is yet more expressive than classical linear logic by giving a simple compositional translation. Because there are fewer symmetries, we obtain a yet again richer set of connectives. For example, ? and ! cannot be defined in terms of each other via negation, echoing a related phenomenon in intuitionistic modal logic [33].

We use a standard two-sided sequent calculus for classical linear logic (CLL), given by the judgment $\Delta \xrightarrow{\text{CLL}} \Omega$ presented in Fig. 3. This formulation allows us to relate CLL and JILL in two different ways: by interpreting CLL in JILL via a uniform translation of propositions, and by showing that JILL is consistent with a unitary restriction of CLL where every right-hand Ω in a deduction is a singleton. As usual, we take exchange in Δ and Ω for granted. We write $!\Delta$ for a collection of assumptions of the form $!A$ and $?\Omega$ for a collection of conclusions of the form $?A$.

CLL enjoys cut-elimination (see [37, 40, 31]), although this property is not needed here. We write weak* and contr* for repeated applications of weakening (weak! and weak?) and contraction rules (contr! and contr?), respectively. Sequents of the form $!\Delta, A \xrightarrow{\text{CLL}} A, ?\Omega$ that follow from weak* and init we justify as init*.

4.1 The intuitionistic fragment of classical linear logic

Gentzen [15] observed that one way to characterize intuitionistic logic is by restricting classical sequents to have an empty or unitary right-hand side, but without changing any of the rules.

$$\frac{}{A \xrightarrow{\text{CLL}} A}\ \text{init} \qquad \frac{\Delta_1 \xrightarrow{\text{CLL}} A, \Omega_1 \quad \Delta_2, A \xrightarrow{\text{CLL}} \Omega_2}{\Delta_1, \Delta_2 \xrightarrow{\text{CLL}} \Omega_1, \Omega_2}\ \text{cut}$$

$$\frac{\Delta, A \xrightarrow{\text{CLL}} \Omega}{\Delta, !A \xrightarrow{\text{CLL}} \Omega}\ !L \qquad \frac{\Delta \xrightarrow{\text{CLL}} \Omega}{\Delta, !A \xrightarrow{\text{CLL}} \Omega}\ \text{weak!} \qquad \frac{\Delta, !A, !A \xrightarrow{\text{CLL}} \Omega}{\Delta, !A \xrightarrow{\text{CLL}} \Omega}\ \text{contr!} \qquad \frac{!\Delta \xrightarrow{\text{CLL}} A, ?\Omega}{!\Delta \xrightarrow{\text{CLL}} !A, ?\Omega}\ !R$$

$$\frac{\Delta \xrightarrow{\text{CLL}} A, \Omega}{\Delta \xrightarrow{\text{CLL}} ?A, \Omega}\ ?R \qquad \frac{\Delta \xrightarrow{\text{CLL}} \Omega}{\Delta \xrightarrow{\text{CLL}} ?A, \Omega}\ \text{weak?} \qquad \frac{\Delta \xrightarrow{\text{CLL}} ?A, ?A, \Omega}{\Delta \xrightarrow{\text{CLL}} ?A, \Omega}\ \text{contr?} \qquad \frac{!\Delta, A \xrightarrow{\text{CLL}} ?\Omega}{!\Delta, ?A \xrightarrow{\text{CLL}} ?\Omega}\ ?L$$

$$\frac{\Delta_1 \xrightarrow{\text{CLL}} A, \Omega_1 \quad \Delta_2 \xrightarrow{\text{CLL}} B, \Omega_2}{\Delta_1, \Delta_2 \xrightarrow{\text{CLL}} A \otimes B, \Omega_1, \Omega_2}\ \otimes R \qquad \frac{\Delta, A, B \xrightarrow{\text{CLL}} \Omega}{\Delta, A \otimes B \xrightarrow{\text{CLL}} \Omega}\ \otimes L$$

$$\frac{}{\cdot \xrightarrow{\text{CLL}} \mathbf{1}}\ \mathbf{1}R \qquad \frac{\Delta \xrightarrow{\text{CLL}} \Omega}{\Delta, \mathbf{1} \xrightarrow{\text{CLL}} \Omega}\ \mathbf{1}L$$

$$\frac{\Delta \xrightarrow{\text{CLL}} A, B, \Omega}{\Delta \xrightarrow{\text{CLL}} A \parr B, \Omega}\ \parr R \qquad \frac{\Delta_1, A \xrightarrow{\text{CLL}} \Omega_1 \quad \Delta_2, B \xrightarrow{\text{CLL}} \Omega_2}{\Delta_1, \Delta_2, A \parr B \xrightarrow{\text{CLL}} \Omega_1, \Omega_2}\ \parr L$$

$$\frac{\Delta \xrightarrow{\text{CLL}} \Omega}{\Delta \xrightarrow{\text{CLL}} \bot, \Omega}\ \bot R \qquad \frac{}{\bot \xrightarrow{\text{CLL}} \cdot}\ \bot L$$

$$\frac{\Delta \xrightarrow{\text{CLL}} A, \Omega \quad \Delta \xrightarrow{\text{CLL}} B, \Omega}{\Delta \xrightarrow{\text{CLL}} A \,\&\, B, \Omega}\ \& R \qquad \frac{\Delta, A \xrightarrow{\text{CLL}} \Omega}{\Delta, A \,\&\, B \xrightarrow{\text{CLL}} \Omega}\ \& L_1 \qquad \frac{\Delta, B \xrightarrow{\text{CLL}} \Omega}{\Delta, A \,\&\, B \xrightarrow{\text{CLL}} \Omega}\ \& L_2$$

$$\frac{}{\Delta \xrightarrow{\text{CLL}} \top, \Omega}\ \top R \qquad \text{no } \top L \text{ rule}$$

$$\frac{\Delta \xrightarrow{\text{CLL}} A, \Omega}{\Delta \xrightarrow{\text{CLL}} A \oplus B, \Omega}\ \oplus R_1 \qquad \frac{\Delta \xrightarrow{\text{CLL}} B, \Omega}{\Delta \xrightarrow{\text{CLL}} A \oplus B, \Omega}\ \oplus R_2 \qquad \frac{\Delta, A \xrightarrow{\text{CLL}} \Omega \quad \Delta, B \xrightarrow{\text{CLL}} \Omega}{\Delta, A \oplus B \xrightarrow{\text{CLL}} \Omega}\ \oplus L$$

$$\text{no } \mathbf{0}R \text{ rule} \qquad \frac{}{\Delta, \mathbf{0} \xrightarrow{\text{CLL}} \Omega}\ \mathbf{0}L$$

$$\frac{\Delta, A \xrightarrow{\text{CLL}} B, \Omega}{\Delta \xrightarrow{\text{CLL}} A \multimap B, \Omega}\ \multimap R \qquad \frac{\Delta_1 \xrightarrow{\text{CLL}} A, \Omega_1 \quad \Delta_2, B \xrightarrow{\text{CLL}} \Omega_2}{\Delta_1, \Delta_2, A \multimap B \xrightarrow{\text{CLL}} \Omega_1, \Omega_2}\ \multimap L$$

$$\frac{\Delta, A \xrightarrow{\text{CLL}} \Omega}{\Delta \xrightarrow{\text{CLL}} \neg A, \Omega}\ \neg R \qquad \frac{\Delta \xrightarrow{\text{CLL}} A, \Omega}{\Delta, \neg A \xrightarrow{\text{CLL}} \Omega}\ \neg L$$

Figure 3: Sequent calculus for classical linear logic CLL

This was generalized by Girard [16] who interpreted intuitionistic logic in classical linear logic, taking advantage of the lack of contraction in the embedding. In this section, we show that intuitionistic linear logic can be characterized in its sequent calculus form as the restriction of sequent derivations in classical linear logic with a unitary right-hand side. This restriction is sometimes taken as a definition of "intuitionistic linear logic"; for example, see Girard *et al.* [18, 19], or Troelstra [40] who additionally permits an empty right-hand side, allowing a definition of $\bot$ and $\neg$. A similar observation was made by Girard for his Logic of Unity (LU), where he characterized intuitionistic linear logic as a right-singleton fragment of a more general language with polarities [17]. We shall identify sequents in the fragment of CLL that have a derivation consisting entirely of unitary sequents—sequents with a single proposition on the right of the arrow—as $\Delta \xrightarrow{\text{ILL}} C$. For these sequents, we allow only the core connectives of JILL, omitting $\neg$, $\bot$, $?$ and $\boxplus$. Note that instances of the core properties of CLL that have a unitary right-hand side remain valid for ILL, including cut, weakening, and contraction as applicable.

Theorem 7 (JILL as Unitary CLL).

1. *If* $!\Gamma, \Delta \xrightarrow{\text{ILL}} C$ *then* $\Gamma\,;\Delta \Longrightarrow C\,;\cdot$
2. *If* $!\Gamma, \Delta \xrightarrow{\text{ILL}} ?C$ *then* $\Gamma\,;\Delta \Longrightarrow \cdot\,;C$
3. *If* $\Gamma\,;\Delta \Longrightarrow C\,;\cdot$ *then* $!\Gamma, \Delta \xrightarrow{\text{ILL}} C$
4. *if* $\Gamma\,;\Delta \Longrightarrow \cdot\,;C$ *then* $!\Gamma, \Delta \xrightarrow{\text{ILL}} ?C$.

Proof. Properties (1) and (2). By simultaneous induction on $\mathcal{D}_1 :: !\Gamma, \Delta \xrightarrow{\text{ILL}} C$ and $\mathcal{D}_2 :: !\Gamma, \Delta \xrightarrow{\text{ILL}} ?C$. Note that rules weak?, contr?, $⅋R$, $⅋L$, $\bot R$, $\bot L$, $\neg R$, and $\neg L$ are not applicable in $\xrightarrow{\text{ILL}}$. We show in detail a few representative cases for the remaining rules.

Case: The last rule in $\mathcal{D}_1$ is init, *i.e.*,

$$\mathcal{D}_1 = \frac{}{A \xrightarrow{\text{ILL}} A}\ \text{init}$$

$$\begin{array}{ll} \cdot\,;A \Longrightarrow A\,;\cdot & \text{by init} \end{array}$$

Case: The last rule in $\mathcal{D}_1$ is $!L$, *i.e.*,

$$\mathcal{D}_1 = \frac{\begin{array}{c}\mathcal{D}_1' \\ !\Gamma, \Delta, A \xrightarrow{\text{ILL}} C\end{array}}{!\Gamma, \Delta, !A \xrightarrow{\text{ILL}} C}\ !L$$

$$\begin{array}{ll}
\Gamma\,;\Delta, A \Longrightarrow C\,;\cdot & \text{by i.h. on } \mathcal{D}_1' \\
\Gamma, A\,;\Delta, A \Longrightarrow C\,;\cdot & \text{by weakening (Thm. 2 (1))} \\
\Gamma, A\,;\Delta \Longrightarrow C\,;\cdot & \text{by copy} \\
\Gamma\,;\Delta, !A \Longrightarrow C\,;\cdot & \text{by } !L
\end{array}$$

The last two lines prove this case, depending on whether $!A$ is added to Δ or $!\Gamma$.

Case: The last rule in $\mathcal{D}_2$ is $?R$, *i.e.*,

$$\mathcal{D}_2 = \frac{\begin{array}{c}\mathcal{D}_2' \\ !\Gamma, \Delta \xrightarrow{\text{ILL}} A\end{array}}{!\Gamma, \Delta \xrightarrow{\text{ILL}} ?A}\ ?R$$

$$\begin{array}{ll}
\Gamma\,;\Delta \Longrightarrow A\,;\cdot & \text{by i.h. on } \mathcal{D}_2' \\
\Gamma\,;\Delta \Longrightarrow \cdot\,;A & \text{by promote} \\
\Gamma\,;\Delta \Longrightarrow ?A\,;\cdot & \text{by } ?R
\end{array}$$

The last two lines prove properties (2) and (1) respectively.

Case: The last rule in $\mathcal{D}_1$ is contr!, *i.e.*,

$$\mathcal{D}_1 = \frac{\begin{array}{c}\mathcal{D}_1' \\ !\Gamma, \Delta, !A, !A \xrightarrow{\text{ILL}} C\end{array}}{!\Gamma, \Delta, !A \xrightarrow{\text{ILL}} C}\ \text{contr}!$$

$$\begin{array}{ll}
\Gamma, A, A\,;\Delta \Longrightarrow C\,;\cdot & \text{by i.h. on } \mathcal{D}_1' \\
\Gamma, A\,;\Delta \Longrightarrow C\,;\cdot & \text{by contraction (Thm. 2 (2))} \\
\Gamma\,;\Delta, !A \Longrightarrow C\,;\cdot & \text{by } !L
\end{array}$$

The last two lines prove this case, depending on whether $!A$ is added to Δ or $!\Gamma$.

Case: The last rule in $\mathcal{D}_1$ is $\otimes R$, *i.e.*,

$$\mathcal{D}_1 = \cfrac{\begin{array}{cc} \mathcal{D}_1' & \mathcal{D}_1'' \\ !\Gamma_1, \Delta_1 \xrightarrow{\text{ILL}} A & !\Gamma_2, \Delta_2 \xrightarrow{\text{ILL}} B \end{array}}{!\Gamma_1, !\Gamma_2, \Delta_1, \Delta_2 \xrightarrow{\text{ILL}} A \otimes B} \otimes R$$

$\Gamma_1 ; \Delta_1 \Longrightarrow A ; \cdot$	by i.h. on $\mathcal{D}_1'$
$\Gamma_1, \Gamma_2 ; \Delta_1 \Longrightarrow A ; \cdot$	by weakening (Thm. 2 (1))
$\Gamma_2 ; \Delta_2 \Longrightarrow B ; \cdot$	by i.h. on $\mathcal{D}_1''$
$\Gamma_1, \Gamma_2 ; \Delta_2 \Longrightarrow B ; \cdot$	by weakening (Thm. 2 (1))
$\Gamma_1, \Gamma_2 ; \Delta_1, \Delta_2 \Longrightarrow A \otimes B ; \cdot$	by $\otimes R$

Other cases are similar.

Properties (3) and (4). By simultaneous induction on $\mathcal{D}_1 :: \Gamma;\Delta \Longrightarrow C;\cdot$ and $\mathcal{D}_2 :: \Gamma;\Delta \Longrightarrow \cdot;C$. For initial sequents, we use init*. In all other cases, we appeal directly to the induction hypothesis and use simple properties of CLL. We show in detail some representative cases.

Case: The last rule in $\mathcal{D}_1$ is init, *i.e.*,

$$\mathcal{D}_1 = \cfrac{}{\Gamma ; A \Longrightarrow A ; \cdot} \text{ init}$$

$!\Gamma, A \xrightarrow{\text{ILL}} A$	by init*

Case: The last rule in $\mathcal{D}_2$ is copy, *i.e.*,

$$\mathcal{D}_2 = \cfrac{\begin{array}{c} \mathcal{D}_2' \\ \Gamma, A ; \Delta, A \Longrightarrow \cdot ; C \end{array}}{\Gamma, A ; \Delta \Longrightarrow \cdot ; C} \text{ copy}$$

$!\Gamma, !A, \Delta, A \xrightarrow{\text{ILL}} ?C$	by i.h. on $\mathcal{D}_2'$
$!\Gamma, !A, \Delta, !A \xrightarrow{\text{ILL}} ?C$	by $!L$
$!\Gamma, !A, \Delta \xrightarrow{\text{ILL}} ?C$	by contr$!$

Case: The last rule in $\mathcal{D}_1$ is $\otimes R$, *i.e.*,

$$\mathcal{D}_1 = \cfrac{\begin{array}{cc} \mathcal{D}_1' & \mathcal{D}_1'' \\ \Gamma ; \Delta_1 \Longrightarrow A & \Gamma ; \Delta_2 \Longrightarrow B \end{array}}{\Gamma ; \Delta_1, \Delta_2 \Longrightarrow A \otimes B ; \cdot} \otimes R$$

$!\Gamma, \Delta_1 \xrightarrow{\text{ILL}} A$	by i.h. on $\mathcal{D}_1'$
$!\Gamma, \Delta_2 \xrightarrow{\text{ILL}} B$	by i.h. on $\mathcal{D}_1''$
$!\Gamma, !\Gamma, \Delta_1, \Delta_2 \xrightarrow{\text{ILL}} A \otimes B$	by $\otimes R$
$!\Gamma, \Delta_1, \Delta_2 \xrightarrow{\text{ILL}} A \otimes B$	by contr*

The remaining cases are similar. □

This theorem can be generalized by allowing the right-hand side to be possibly empty, and using the extension of JILL with a possibly empty right-hand side as in Sec. 2.4. This allows us to consider $\perp$ and $\neg$ directly; a similar observation was made by Troelstra for the system ILZ [40]. We can generalize even further by allowing the right-hand side to be either a collection $?\Omega$ or a *single* proposition C; we then obtain the multi-conclusion formulation of JILL in Sec. 2.5 as a right-restricted form of CLL. The CLL connective $⅋$ can now be allowed in the restricted form $?A ⅋ ?B$; note that $?A ⅋ ?B \overset{\text{CLL}}{\equiv} ?(A \oplus B)$, so we recover our modal disjunction. The technical details of these extensions are entirely straightforward and omitted.

4.2 Double negation translation

The intuitionist views a classical proof of A as a refutation of its negation. In our setting, this would correspond to a proof of $\cdot\,;\neg [\![A]\!] \Longrightarrow \bot\,;\cdot$, where $\neg [\![A]\!]$ is the negation of the translation of A. It is more economical and allows some further applications if we instead parameterize the translation by a propositional parameter p. This idea is due to Friedman [14] in the case of ordinary classical and intuitionistic logic, and was also employed by Lamarche [25] in the setting of game semantics. It is convenient to introduce a *parametric negation* $\sim_p A$, defined to be $A \multimap p$; the usual linear negation in JILL thus corresponds to $\sim_\bot A$. The translation is compositional. For most connectives ($\otimes$, $\mathbf{1}$, $\&$, $\top$, $\oplus$, $\mathbf{0}$, $!$, $\multimap$) we prefix each subformula with a double negation, though some optimizations are possible. Negation ($\neg$) is translated simply to the parametric negation $\sim_p$. The remaining connectives ($\invamp$, $\bot$, $?$) are translated by exploiting classical equivalences. For example, we think of $\bot$ as $\neg\mathbf{1}$. Together with the use of the parameter p, this means we do not need to employ $?$, $\bot$, $\neg$, or $\boxplus$ in the image of the translation.

$$
\begin{array}{rlrl}
[\![P]\!]_p &= P & & \\
[\![A \otimes B]\!]_p &= \sim_p\sim_p [\![A]\!]_p \otimes \sim_p\sim_p [\![B]\!]_p & [\![\mathbf{1}]\!]_p &= \mathbf{1} \\
[\![A \invamp B]\!]_p &= \sim_p(\sim_p [\![A]\!]_p \otimes \sim_p [\![B]\!]_p) & [\![\bot]\!]_p &= \sim_p\mathbf{1} \\
[\![A \,\&\, B]\!]_p &= \sim_p\sim_p [\![A]\!]_p \,\&\, \sim_p\sim_p [\![B]\!]_p & [\![\top]\!]_p &= \top \\
[\![A \oplus B]\!]_p &= \sim_p\sim_p [\![A]\!]_p \oplus \sim_p\sim_p [\![B]\!]_p & [\![\mathbf{0}]\!]_p &= \mathbf{0} \\
[\![!A]\!]_p &= !\sim_p\sim_p [\![A]\!]_p & [\![?A]\!]_p &= \sim_p!\sim_p [\![A]\!]_p \\
[\![A \multimap B]\!]_p &= \sim_p\sim_p [\![A]\!]_p \multimap \sim_p\sim_p [\![B]\!]_p & [\![\neg A]\!]_p &= \sim_p [\![A]\!]
\end{array}
$$

We lift this definition to contexts of propositions by translating every proposition in the context.

Lemma 8.

1. *If* $\Gamma\,;\Delta \Longrightarrow A\,;\cdot$, *then* $\Gamma\,;\Delta, \sim_p A \Longrightarrow p\,;\cdot$.
2. $\Gamma\,;\Delta, A \Longrightarrow p\,;\cdot$ *iff* $\Gamma\,;\Delta, \sim_p\sim_p A \Longrightarrow p\,;\cdot$.

Proof. For (1), use $\multimap L$ with the given sequent and the initial sequent $\Gamma\,;p \Longrightarrow p\,;\cdot$ as premisses. For (2), in the forward direction use $\multimap R$ and part (1); in the reverse direction, note that $\Gamma\,;A \Longrightarrow \sim_p\sim_p A\,;\cdot$ is derivable:

$$
\dfrac{\dfrac{\dfrac{}{\Gamma\,;A \Longrightarrow A\,;\cdot}\ \text{init}}{\Gamma\,;A, \sim_p A \Longrightarrow p\,;\cdot}\ \text{part (1)}}{\Gamma\,;A \Longrightarrow \sim_p\sim_p A\,;\cdot}\ \multimap R
$$

Then apply cut (Thm. 3) with cut-formula $\sim_p\sim_p A$. □

Given a collection of formulas Δ, we represent by $\sim_p\Delta$ the corresponding collection obtained by prefixing every formula in Δ with $\sim_p$.

Theorem 9 (Preservation). *If* $\Delta \xrightarrow{\text{CLL}} \Omega$, *then for any proposition* p, *we have* $\cdot\,;[\![\Delta]\!]_p\,, \sim_p [\![\Omega]\!]_p \Longrightarrow p;\cdot$.

Proof. By structural induction on the derivation $\mathcal{C} :: \Delta \xrightarrow{\text{CLL}} \Omega$. In some cases we use the observation that $\Gamma\,;\Delta \Longrightarrow \gamma$ iff $\cdot\,;!\Gamma, \Delta \Longrightarrow \gamma$. This follows by repeated applications of $!L$ in one direction and by repeated application of $!L^{-1}$ (Lem. 6) in the other direction; we refer to these properties as $!L^*$ and $!L^{*-1}$. We highlight a few representative cases.

Case: The last rule in $\mathcal{C}$ is init, *i.e.*,

$$\mathcal{C} = \frac{}{A \xrightarrow{\text{CLL}} A}\ \text{init}$$

$\cdot\,;\, [\![A]\!]_p \Longrightarrow [\![A]\!]_p\,;\,\cdot$ by init
$\cdot\,;\, [\![A]\!]_p\,, \sim_p [\![A]\!]_p \Longrightarrow p\,;\,\cdot$ by Lem. 8 (1)

Case: The last rule in $\mathcal{C}$ is cut, *i.e.*,

$$\mathcal{C} = \frac{\begin{matrix}\mathcal{C}' \\ \Delta_1 \xrightarrow{\text{CLL}} A, \Omega_1\end{matrix} \quad \begin{matrix}\mathcal{C}'' \\ \Delta_2, A \xrightarrow{\text{CLL}} \Omega_2\end{matrix}}{\Delta_1, \Delta_2 \xrightarrow{\text{CLL}} \Omega_1, \Omega_2}\ \text{cut}$$

$\cdot\,;\, [\![\Delta_1]\!]_p\,, \sim_p [\![A]\!]_p\,, \sim_p [\![\Omega_1]\!]_p \Longrightarrow p\,;\,\cdot$ by i.h. on $\mathcal{C}'$
$\cdot\,;\, [\![\Delta_2]\!]_p\,, [\![A]\!]_p\,, \sim_p [\![\Omega_2]\!]_p \Longrightarrow p\,;\,\cdot$ by i.h. on $\mathcal{C}''$
$\cdot\,;\, [\![\Delta_2]\!]_p\,, \sim_p [\![\Omega_2]\!]_p \Longrightarrow \sim_p [\![A]\!]_p\,;\,\cdot$ by $\multimap R$
$\cdot\,;\, [\![\Delta_1]\!]_p\,, [\![\Delta_2]\!]_p\,, \sim_p [\![\Omega_1]\!]_p\,, \sim_p [\![\Omega_2]\!]_p \Longrightarrow p\,;\,\cdot$ by cut (Thm. 3)

Case: The last rule in $\mathcal{C}$ is a weakening rule, say weak!.

$$\mathcal{C} = \frac{\begin{matrix}\mathcal{C}' \\ \Delta \xrightarrow{\text{CLL}} \Omega\end{matrix}}{\Delta, !A \xrightarrow{\text{CLL}} \Omega}\ \text{weak!}$$

$\cdot\,;\, [\![\Delta]\!]_p\,, \sim_p [\![\Omega]\!]_p \Longrightarrow p\,;\,\cdot$ by i.h. on $\mathcal{C}'$
$\sim_p\sim_p [\![A]\!]_p\,;\, [\![\Delta]\!]_p\,, \sim_p [\![\Omega]\!]_p \Longrightarrow p\,;\,\cdot$ by weakening (Thm. 2 (1))
$\cdot\,;\, [\![!A]\!]_p\,, [\![\Delta]\!]_p\,, \sim_p [\![\Omega]\!]_p \Longrightarrow p\,;\,\cdot$ by $!L$

Case: The last rule in $\mathcal{C}$ is a contraction rule, say contr!.

$$\mathcal{C} = \frac{\begin{matrix}\mathcal{C}' \\ \Delta, !A, !A \xrightarrow{\text{CLL}} \Omega\end{matrix}}{\Delta, !A \xrightarrow{\text{CLL}} \Omega}\ \text{contr!}$$

$\cdot\,;\, [\![\Delta]\!]_p\,, [\![!A]\!]_p\,, [\![!A]\!]_p\,, \sim_p [\![\Omega]\!]_p \Longrightarrow p\,;\,\cdot$ by i.h. on $\mathcal{C}'$
$\sim_p\sim_p [\![A]\!]_p\,, \sim_p\sim_p [\![A]\!]_p\,;\, [\![\Delta]\!]_p\,, \sim_p [\![\Omega]\!]_p \Longrightarrow p\,;\,\cdot$ by $!L^{*-1}$
$\sim_p\sim_p [\![A]\!]_p\,;\, [\![\Delta]\!]_p\,, \sim_p [\![\Omega]\!]_p \Longrightarrow p\,;\,\cdot$ by contraction (Thm. 2 (2))
$\cdot\,;\, [\![\Delta]\!]_p\,, [\![!A]\!]_p\,, \sim_p [\![\Omega]\!]_p \Longrightarrow p\,;\,\cdot$ by $!L$

Case: The last rule in $\mathcal{C}$ is a multiplicative rule, say $\otimes R$.

$$\mathcal{C} = \frac{\begin{matrix}\mathcal{C}' \\ \Delta_1 \xrightarrow{\text{CLL}} A, \Omega_1\end{matrix} \quad \begin{matrix}\mathcal{C}'' \\ \Delta_2 \xrightarrow{\text{CLL}} B, \Omega_2\end{matrix}}{\Delta_1, \Delta_2 \xrightarrow{\text{CLL}} A \otimes B, \Omega_1, \Omega_2}\ \otimes R$$

$\cdot\,;\, [\![\Delta_1]\!]_p\,, \sim_p [\![A]\!]_p\,, \sim_p [\![\Omega_1]\!]_p \Longrightarrow p\,;\,\cdot$ by i.h. on $\mathcal{C}'$
$\cdot\,;\, [\![\Delta_1]\!]_p\,, \sim_p [\![\Omega_1]\!]_p \Longrightarrow \sim_p\sim_p [\![A]\!]_p\,;\,\cdot$ by $\multimap R$
$\cdot\,;\, [\![\Delta_1]\!]_p\,, \sim_p [\![B]\!]_p\,, \sim_p [\![\Omega_1]\!]_p \Longrightarrow p\,;\,\cdot$ by i.h. on $\mathcal{C}''$
$\cdot\,;\, [\![\Delta_2]\!]_p\,, \sim_p [\![\Omega_2]\!]_p \Longrightarrow \sim_p\sim_p [\![B]\!]_p\,;\,\cdot$ by $\multimap R$
$\cdot\,;\, [\![\Delta_1, \Delta_2]\!]_p\,, \sim_p [\![\Omega_1, \Omega_2]\!]_p \Longrightarrow [\![A \otimes B]\!]_p\,;\,\cdot$ by $\otimes R$
$\cdot\,;\, [\![\Delta_1, \Delta_2]\!]_p\,, \sim_p [\![\Omega_1, \Omega_2]\!]_p\,, \sim_p [\![A \otimes B]\!]_p \Longrightarrow p\,;\,\cdot$ by Lem. 8 (1)

Case: The last rule in $\mathcal{C}$ is $\bot R$, *i.e.*,

$$\mathcal{C} = \dfrac{\begin{matrix}\mathcal{C}' \\ \Delta \xrightarrow{\text{CLL}} \Omega\end{matrix}}{\Delta \xrightarrow{\text{CLL}} \bot, \Omega} \ \bot R$$

$\cdot\,;\, [\![\Delta]\!]_p\,, {\sim_p}\, [\![\Omega]\!]_p \Longrightarrow p\,;\,\cdot$	by i.h. on $\mathcal{C}'$
$\cdot\,;\, [\![\Delta]\!]_p\,, {\sim_p}\, [\![\Omega]\!]_p\,, \mathbf{1} \Longrightarrow p\,;\,\cdot$	by $\mathbf{1}L$
$\cdot\,;\, [\![\Delta]\!]_p\,, {\sim_p}\, [\![\Omega]\!]_p \Longrightarrow [\![\bot]\!]_p\,;\,\cdot$	by $\multimap R$
$\cdot\,;\, [\![\Delta]\!]_p\,, {\sim_p}\, [\![\Omega]\!]_p\,, {\sim_p}\, [\![\bot]\!]_p \Longrightarrow p\,;\,\cdot$	by Lem. 8 (1)

Case: The last rule in $\mathcal{C}$ is $!R$, *i.e.*,

$$\mathcal{C} = \dfrac{\begin{matrix}\mathcal{C}' \\ !\Gamma \xrightarrow{\text{CLL}} A, ?\Omega\end{matrix}}{!\Gamma \xrightarrow{\text{CLL}} !A, ?\Omega} \ !R$$

$\cdot\,;\, [\![!\Gamma]\!]_p\,, {\sim_p}\, [\![?\Omega]\!]_p\,, {\sim_p}\, [\![A]\!]_p \Longrightarrow p\,;\,\cdot$	by i.h. on $\mathcal{C}'$
$\cdot\,;\, [\![!\Gamma]\!]_p\,, !{\sim_p}\, [\![\Omega]\!]_p\,, {\sim_p}\, [\![A]\!]_p \Longrightarrow p\,;\,\cdot$	by repeated Lem. 8 (2)
${\sim_p}{\sim_p}\, [\![\Gamma]\!]_p\,, {\sim_p}\, [\![\Omega]\!]_p\,;\, {\sim_p}\, [\![A]\!]_p \Longrightarrow p\,;\,\cdot$	by $!L^{*-1}$
${\sim_p}{\sim_p}\, [\![\Gamma]\!]_p\,, {\sim_p}\, [\![\Omega]\!]_p\,;\,\cdot \Longrightarrow {\sim_p}{\sim_p}\, [\![A]\!]_p\,;\,\cdot$	by $\multimap R$
${\sim_p}{\sim_p}\, [\![\Gamma]\!]_p\,, {\sim_p}\, [\![\Omega]\!]_p\,;\,\cdot \Longrightarrow [\![!A]\!]_p\,;\,\cdot$	by $!R$
${\sim_p}{\sim_p}\, [\![\Gamma]\!]_p\,, {\sim_p}\, [\![\Omega]\!]_p\,;\, {\sim_p}\, [\![!A]\!]_p \Longrightarrow p\,;\,\cdot$	by Lem. 8 (1)
$\cdot\,;\, [\![!\Gamma]\!]_p\,, !{\sim_p}\, [\![\Omega]\!]_p\,, {\sim_p}\, [\![!A]\!]_p \Longrightarrow p\,;\,\cdot$	by $!L^*$
$\cdot\,;\, [\![!\Gamma]\!]_p\,, {\sim_p}\, [\![?\Omega]\!]_p\,, {\sim_p}\, [\![!A]\!]_p \Longrightarrow p\,;\,\cdot$	by repeated Lem. 8 (2)

□

Next we need to establish the *soundness* of the translation, *i.e.*, that JILL provability of formulas in the image of this translation guarantees CLL-provability of the corresponding source CLL formulas. Such a statement cannot be shown by induction on the derivation of $[\![\Gamma]\!]_p;[\![\Delta]\!]_p \Longrightarrow p;\cdot$ because the JILL derivation might refer to propositions that do not exist in the image of the translation. Instead, we make the simple observation that classical linear logic admits more proofs on the same connectives: the translation in the reverse direction, from JILL to CLL, is simply the identity.

Corollary 10 (JILL proofs as CLL proofs).

1. *If* $\Gamma\,;\,\Delta \Longrightarrow A\,;\,\cdot$, *then* $!\Gamma, \Delta \xrightarrow{\mathit{CLL}} A$.
2. *If* $\Gamma\,;\,\Delta \Longrightarrow \cdot\,;\,A$, *then* $!\Gamma, \Delta \xrightarrow{\mathit{CLL}} ?A$.

Proof. Directly from Thm. 7 (3,4), by noting that any ILL proof is also a CLL proof. □

This interpretation of JILL proofs as CLL proofs that treats JILL formulas as the corresponding identical CLL formulas gives an inverse of the $[\![-]\!]_p$ translation up to classical equivalence.

Lemma 11. *For any proposition* A, $[\![A]\!]_\bot \overset{\mathit{CLL}}{\equiv} A$.

Proof. By structural induction on the proposition A, using the observation that for any proposition B, ${\sim_\bot} B \overset{\text{CLL}}{\equiv} \neg B$, and noting that $\neg\neg B \overset{\text{CLL}}{\equiv} B$. The interesting cases are the ones where the $[\![-]\!]_p$ translation does not double-negate the operands:

$$[\![B \parr C]\!]_\bot = {\sim_\bot}({\sim_\bot}\, [\![B]\!]_\bot \otimes {\sim_\bot}\, [\![C]\!]_\bot) \overset{\text{CLL}}{\equiv} \neg(\neg\, [\![B]\!]_\bot \otimes \neg\, [\![C]\!]_\bot) \overset{\text{CLL}}{\equiv} \neg\neg\, [\![B]\!]_\bot \parr \neg\neg\, [\![C]\!]_\bot \overset{\text{CLL}}{\equiv} [\![B]\!]_\bot \parr [\![C]\!]_\bot$$

$$[\![\bot]\!]_\bot = \sim_\bot \mathbf{1} \overset{\text{CLL}}{\equiv} \neg\mathbf{1} \overset{\text{CLL}}{\equiv} \bot$$
$$[\![?A]\!]_\bot = \sim_\bot !\sim_\bot [\![A]\!]_\bot \overset{\text{CLL}}{\equiv} \neg!\neg [\![A]\!]_\bot \overset{\text{CLL}}{\equiv} ?[\![A]\!]_\bot$$

The cases for the other connectives are obvious. ☐

Theorem 12. *In any extension of CLL (referred to as CLL$^+$) where $C \overset{CLL^+}{\equiv} \bot$, if $[\![\Delta]\!]_C, \sim_C [\![\Omega]\!]_C \xrightarrow{CLL^+} C$, then $\Delta \xrightarrow{CLL^+} \Omega$.*

Proof. Using Lem. 11, we get $\Delta, \sim_\bot \Omega \xrightarrow{\text{CLL}^+} \bot$. We then use cut with formula $\bot$ and the evident sequent $\bot \xrightarrow{\text{CLL}^+} \cdot$ to get $\Delta, \sim_\bot \Omega \xrightarrow{\text{CLL}^+} \cdot$. Since $\sim_\bot A \overset{\text{CLL}}{\equiv} \neg A$, we thus have $\Delta, \neg\Omega \Longrightarrow \cdot$. Now we repeatedly cut with evident sequents of the form $\cdot \xrightarrow{\text{CLL}} A, \neg A$ where the propositions A are drawn from Ω. ☐

The soundness theorem follows directly.

Corollary 13 (Soundness). *If $\cdot\,; [\![\Delta]\!]_p \Longrightarrow p\,; \cdot$ for a propositional parameter p, then $\Delta \xrightarrow{CLL} \cdot$.*

Proof. Instantiate p with $\bot$. From Cor. 10, we know $[\![\Delta]\!]_\bot \xrightarrow{\text{CLL}} \bot$, so we use Thm. 12. ☐

4.3 Exploiting parametricity

By design, the $[\![-]\!]_p$ translation works even in the absence of linear contradiction $\bot$ on the intuitionistic side, but as shown in Sec. 2.4, JILL allows a definition of $\bot$ as $?\mathbf{0}$. Using this definition for the parameter p, we obtain an elegant characterization of CLL.

Theorem 14 (Characterizing CLL). $\Delta \xrightarrow{CLL} \Omega$ *iff* $\cdot\,; [\![\Delta]\!]_\bot, \sim_\bot [\![\Omega]\!]_\bot \Longrightarrow \bot\,; \cdot$.

Proof. In the forward direction, we use Thm. 9 with the choice $?\mathbf{0}$ ($= \bot$) for the parameter p. For the reverse direction, we use Cor. 10 to conclude $[\![\Delta]\!]_{?\mathbf{0}}, \sim_p [\![\Omega]\!]_{?\mathbf{0}} \xrightarrow{\text{CLL}} ?\mathbf{0}$; then note that $?\mathbf{0} \overset{\text{CLL}}{\equiv} \bot$, so the result follows from Thm. 12. ☐

In other words, a classical (CLL) proof may be viewed as an intuitionistic refutation in JILL. Not all choices for p give the same behavior. For example, the choice $\mathbf{1}$ for p interprets an extension of CLL with the following additional MIX rules, which were first considered by Girard [16].

$$\frac{}{\cdot \xrightarrow{\text{CLL}} \cdot}\ \text{MIX}_0 \qquad \frac{\Delta_1 \xrightarrow{\text{CLL}} \Omega_1 \quad \Delta_2 \xrightarrow{\text{CLL}} \Omega_2}{\Delta_1, \Delta_2 \xrightarrow{\text{CLL}} \Omega_1, \Omega_2}\ \text{MIX}_2$$

We write CLL^{0,2} for CLL with both MIX rules. Like Thm. 14, $[\![-]\!]_\mathbf{1}$ characterizes CLL^{0,2}.

Theorem 15 (Characterizing CLL^{0,2}). $\Delta \xrightarrow{CLL^{0,2}} \Omega$ *iff* $\cdot\,; [\![\Delta]\!]_\mathbf{1}, \sim_\mathbf{1} [\![\Omega]\!]_\mathbf{1} \Longrightarrow \mathbf{1}\,; \cdot$.

Proof. For the forward direction, we proceed by structural induction on the derivation of $\Delta \xrightarrow{\text{CLL}^{0,2}} \Omega$. The cases for the CLL rules are completely analogous to those of Thm. 9. For the MIX$_0$ case, we use $\mathbf{1}R$, while for the MIX$_2$, we note the admissibility of the following rule in JILL, by using $\mathbf{1}L$ on the second premiss and admissibility of cut (Thm. 3).

$$\frac{\cdot\,; \Delta \Longrightarrow \mathbf{1}\,; \cdot \quad \cdot\,; \Delta' \Longrightarrow \mathbf{1}\,; \cdot}{\cdot\,; \Delta, \Delta' \Longrightarrow \mathbf{1}\,; \cdot}$$

For the reverse direction, we note that in the presence of the MIX rules, $\mathbf{1} \stackrel{\text{CLL}^{0,2}}{\equiv} \bot$:

$$\dfrac{\dfrac{\dfrac{}{\cdot \xrightarrow{\text{CLL}^{0,2}} \cdot}\ \text{MIX}_0}{\mathbf{1} \xrightarrow{\text{CLL}^{0,2}} \cdot}\ \mathbf{1}L}{\mathbf{1} \xrightarrow{\text{CLL}^{0,2}} \bot}\ \bot R \qquad \dfrac{\dfrac{}{\bot \xrightarrow{\text{CLL}^{0,2}} \cdot}\ \bot L \quad \dfrac{}{\cdot \xrightarrow{\text{CLL}^{0,2}} \mathbf{1}}\ \mathbf{1}R}{\bot \xrightarrow{\text{CLL}^{0,2}} \mathbf{1}}\ \text{MIX}_2$$

Therefore, we just use Cor. 10 and Thm. 12. □

Therefore, a proof in $\text{CLL}^{0,2}$ corresponds to a JILL proof of $\mathbf{1}$. In other words, a classical proof using the MIX rules can be seen as an intuitionistic proof that can consume all resources. This analysis interprets $\neg A$ as a *consumer* of A, that is, $A \multimap \mathbf{1}$. This provides a clear understanding of the MIX rules from a logical and constructive viewpoint. Note that, classically, this explanation does not work, because $(A \multimap \mathbf{1}) \multimap \mathbf{1}$ is *not* equivalent to A.

The remarkable uniformity of the parametric translation $[\![-]\!]_p$ demands the question: what about other choices for p? A full examination of all formulas lies beyond the scope of this paper; we present below the cases for the various propositional constants and modal connectives. There are exactly seven distinguishable modal operators – none, !, ?, !?, ?!, !?! and ?!? (see [38] for a more complete discussion); for the propositional constants, there are some further equivalences:

$$\begin{array}{lll} (1)\ ?\top \equiv \top & (2)\ \mathbf{1} \equiv\ !\top \equiv\ !?\top \equiv\ !?!\top & (3)\ ?\mathbf{1} \equiv\ ?!\top \equiv\ ?!?\top \\ (4)\ \mathbf{0} \equiv\ !\mathbf{0} & (5)\ \bot =\ ?\mathbf{0} \equiv\ ?!\mathbf{0} \equiv\ ?!?\mathbf{0} & (6)\ !\bot \equiv\ !?\mathbf{0} \equiv\ !?!\mathbf{0} \end{array}$$

We have already considered the classes 2 and 5. The following are the remaining classes.

4.3.1 The case of $p = \top$ (class 1)

This choice turns out to be uninteresting, because it causes the constants $\top$ and $\bot$ to collapse in the preimage of the translation, giving an inconsistent logic.

4.3.2 The case of $p = ?\mathbf{1}$ (class 3)

This choice gives us CLL with MIX_0 *but not* MIX_2; we refer to this system as CLL^0.

Theorem 16 (Characterizing CLL^0). $\Delta \xrightarrow{\text{CLL}^0} \Omega$ *iff* $\cdot\,;[\![\Delta]\!]_{?\mathbf{1}}, \sim_{?\mathbf{1}}[\![\Omega]\!]_{?\mathbf{1}} \Longrightarrow ?\mathbf{1}\,; \cdot$.

Proof. For the forward direction, we use preservation (Thm. 9) for all rules of CLL and use the evident JILL sequent $\cdot\,;\cdot \Longrightarrow ?\mathbf{1}\,;\cdot$ for the MIX_0 case. For the reverse direction, we note that $?\mathbf{1} \stackrel{\text{CLL}^0}{\equiv} \bot$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot \xrightarrow{\text{CLL}^0} \cdot}\ \text{MIX}_0}{\mathbf{1} \xrightarrow{\text{CLL}^0} \cdot}\ \mathbf{1}L}{?\mathbf{1} \xrightarrow{\text{CLL}^0} \cdot}\ ?L}{?\mathbf{1} \xrightarrow{\text{CLL}^0} \bot}\ \bot R \qquad \dfrac{\dfrac{}{\bot \xrightarrow{\text{CLL}^0} \cdot}\ \bot L}{\bot \xrightarrow{\text{CLL}^0} ?\mathbf{1}}\ \text{weak}?$$

Now we use Cor. 10 and Thm. 12. □

4.3.3 The case of $p = \mathbf{0}$ (class 4).

This choice interprets CLL with additional weakening rules, also known as affine logic (CAL).

$$\frac{\Delta \xrightarrow{\text{CAL}} \Omega}{\Delta, A \xrightarrow{\text{CAL}} \Omega}\ \text{weak}_L \qquad \frac{\Delta \xrightarrow{\text{CAL}} \Omega}{\Delta \xrightarrow{\text{CAL}} A, \Omega}\ \text{weak}_R$$

Theorem 17 (Characterizing CAL). $\Delta \xrightarrow{CAL} \Omega$ *iff* $\cdot\,;\, [\![\Delta]\!]_{\mathbf{0}}\,, \sim_{\mathbf{0}} [\![\Omega]\!]_{\mathbf{0}} \Longrightarrow \mathbf{0}\,;\,\cdot$.

Proof. For the forward direction, we proceed by structural induction on the derivation of $\Delta \xrightarrow{\text{CAL}} \Omega$. The cases for the CLL rules are completely analogous to those of Thm. 9. For weak_L and weak_R, we use the following admissible rule in JILL:

$$\frac{\cdot\,;\,\Delta \Longrightarrow \mathbf{0}\,;\,\cdot}{\cdot\,;\,\Delta, A \Longrightarrow \mathbf{0}\,;\,\cdot}$$

which we obtain using cut-admissibility (Thm. 3) with cut-formula **0** and the evident sequent $\Gamma\,;\,A, \mathbf{0} \Longrightarrow \mathbf{0}\,;\,\cdot$. In the reverse direction, we note that $\bot \stackrel{\text{CAL}}{\equiv} \mathbf{0}$, so Cor. 10 and Thm. 12 gives us the required result. ☐

4.3.4 The case of $p = \mathbf{!}\bot$ (class 6).

The judgmental reconstruction of this system as an extension of CLL that allows the theorem $!\bot \equiv \bot$, remains open. Nevertheless, we can characterize this mysterious system, which we call $\text{CLL}^?$, in a completely analogous manner as the previous theorems. $\text{CLL}^?$ has the remarkable property that MIX_2 is derivable.

$$\dfrac{\dfrac{\dfrac{\Delta_1 \xrightarrow{\text{CLL}^?} \Omega_1}{\Delta_1 \xrightarrow{\text{CLL}^?} \bot, \Omega_1}\ \bot R}{\Delta_1 \xrightarrow{\text{CLL}^?} !\bot, \Omega_1}\ \bot \equiv !\bot \qquad \dfrac{\Delta_2 \xrightarrow{\text{CLL}^?} \Omega_2}{!\bot, \Delta_2 \xrightarrow{\text{CLL}^?} \Omega_2}\ \text{weak!}}{\Delta_1, \Delta_2 \xrightarrow{\text{CLL}^?} \Omega_1, \Omega_2}\ \text{cut}$$

5 Conclusion

We have presented a natural deduction system for intuitionistic linear logic with a clear distinction between judgments and propositions. Besides the usual connectives of intuitionistic linear logic ($\otimes$, $\mathbf{1}$, $\multimap$, $\&$, $\top$, $\oplus$, $\mathbf{0}$, $!$) we also defined $?$, $\bot$, and $\neg$, either via introduction and elimination rules and or as notational definitions. The multiplicative disjunction $⅋$ from classical linear logic also has a counterpart in a modal disjunction $\boxplus$. The judgmental treatment allows the development of a sequent calculus which formalizes Martin-Löf's notion of *verification* as cut-free proofs in the sequent calculus. The sequent calculus has a natural bottom-up reading, by design, and we proved the admissibility of cut by structural induction in a straightforward manner. Using our sequent calculus, we developed a double-negation interpretation of classical linear logic that also accounts for certain extensions of CLL; in particular, we obtained a constructive interpretation of the structural MIX rules.

Term calculi for the natural deduction formulation of JILL and their computational interpretation provides an interesting area for future work. For example, a natural interpretation of $A \multimap ?B$ allows linear partial functions, while $\boxplus$ and $\bot$ may be related to concurrency and continuations. Finally, the nature of possibility requires a categorical explanation as a dual of the standard comonad construction for !.

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A Multiple conclusion presentation of JILL

A.1 Natural deduction

$$\underbrace{A_1 \ \mathit{valid}, \ldots, A_k \ \mathit{valid}}_{\Gamma} \,;\, \underbrace{B_1 \ \mathit{true}, \ldots, B_l \ \mathit{true}}_{\Delta} \vdash \underbrace{\begin{cases} C \ \mathit{true} \\ \underbrace{(C_1 \ \mathit{poss} \mid \cdots \mid C_m \ \mathit{poss})}_{\Sigma} \end{cases}}_{J}$$

Judgmental labels are omitted when understood.

Judgmental Rules

$$\frac{}{\Gamma \,;\, A \vdash A \ \mathit{true}}\ \mathrm{hyp} \qquad \frac{}{\Gamma, A \,;\, \cdot \vdash A \ \mathit{true}}\ \mathrm{hyp!} \qquad \frac{\Gamma \,;\, \Delta \vdash A \ \mathit{true}}{\Gamma \,;\, \Delta \vdash \Sigma \mid A}\ \mathrm{poss}$$

Multiplicative Connectives

$$\frac{\Gamma \,;\, \Delta \vdash A \quad \Gamma \,;\, \Delta' \vdash B}{\Gamma \,;\, \Delta, \Delta' \vdash A \otimes B}\ \otimes I \qquad \frac{\Gamma \,;\, \Delta \vdash A \otimes B \quad \Gamma \,;\, \Delta', A, B \vdash J}{\Gamma \,;\, \Delta, \Delta' \vdash J}\ \otimes E$$

$$\frac{}{\Gamma \,;\, \cdot \vdash \mathbf{1}}\ \mathbf{1}I \qquad \frac{\Gamma \,;\, \Delta \vdash \mathbf{1} \quad \Gamma \,;\, \Delta' \vdash J}{\Gamma \,;\, \Delta, \Delta' \vdash J}\ \mathbf{1}E$$

$$\frac{\Gamma \,;\, \Delta \vdash A \ \mathit{poss} \mid B \ \mathit{poss}}{\Gamma \,;\, \Delta \vdash A \boxplus B}\ \boxplus I \qquad \frac{\Gamma \,;\, \Delta \vdash A \boxplus B \quad \Gamma \,;\, A \vdash \Sigma \quad \Gamma \,;\, B \vdash \Sigma}{\Gamma \,;\, \Delta \vdash \Sigma}\ \boxplus E$$

$$\frac{\Gamma \,;\, \Delta \vdash \cdot}{\Gamma \,;\, \Delta \vdash \bot}\ \bot I \qquad \frac{\Gamma \,;\, \Delta \vdash \bot}{\Gamma \,;\, \Delta \vdash \Sigma}\ \bot E$$

$$\frac{\Gamma\,;\Delta, A \vdash \cdot}{\Gamma\,;\Delta \vdash \neg A}\ \neg I \qquad \frac{\Gamma\,;\Delta \vdash \neg A \quad \Gamma\,;\Delta' \vdash A}{\Gamma\,;\Delta, \Delta' \vdash \Sigma}\ \neg E$$

$$\frac{\Gamma\,;\Delta, A \vdash B}{\Gamma\,;\Delta \vdash A \multimap B}\ \multimap I \qquad \frac{\Gamma\,;\Delta \vdash A \multimap B \quad \Gamma\,;\Delta' \vdash A}{\Gamma\,;\Delta, \Delta' \vdash B}\ \multimap E$$

Additive Connectives

$$\frac{\Gamma\,;\Delta \vdash A \quad \Gamma\,;\Delta \vdash B}{\Gamma\,;\Delta \vdash A \,\&\, B}\ \& I \qquad \frac{\Gamma\,;\Delta \vdash A \,\&\, B}{\Gamma\,;\Delta \vdash A}\ \& E_1 \qquad \frac{\Gamma\,;\Delta \vdash A \,\&\, B}{\Gamma\,;\Delta \vdash B}\ \& E_2$$

$$\frac{}{\Gamma\,;\Delta \vdash \top}\ \top I \qquad \frac{\Gamma\,;\Delta \vdash \mathbf{0}}{\Gamma\,;\Delta, \Delta' \vdash J}\ \mathbf{0}E$$

$$\frac{\Gamma\,;\Delta \vdash A}{\Gamma\,;\Delta \vdash A \oplus B}\ \oplus I_1 \qquad \frac{\Gamma\,;\Delta \vdash B}{\Gamma\,;\Delta \vdash A \oplus B}\ \oplus I_2 \qquad \frac{\Gamma\,;\Delta \vdash A \oplus B \quad \Gamma\,;\Delta', A \vdash J \quad \Gamma\,;\Delta', B \vdash J}{\Gamma\,;\Delta, \Delta' \vdash J}\ \oplus E$$

Exponentials

$$\frac{\Gamma\,;\cdot \vdash A}{\Gamma\,;\cdot \vdash\, !A}\ !I \qquad \frac{\Gamma\,;\Delta \vdash\, !A \quad \Gamma, A\,;\Delta' \vdash J}{\Gamma\,;\Delta, \Delta' \vdash J}\ !E \qquad \frac{\Gamma\,;\Delta \vdash A\ \mathit{poss}}{\Gamma\,;\Delta \vdash\, ?A}\ ?I \qquad \frac{\Gamma\,;\Delta \vdash\, ?A \quad \Gamma\,;A \vdash \Sigma}{\Gamma\,;\Delta \vdash \Sigma}\ ?E$$

A.2 Sequent calculus

$$\underbrace{A_1\ \mathit{valid}, \ldots, A_k\ \mathit{valid}}_{\Gamma}\,;\underbrace{B_1\ \mathit{res}, \ldots, B_l\ \mathit{res}}_{\Delta} \Longrightarrow \underbrace{\begin{cases} C\ \mathit{goal}\,;\cdot \\ \cdot\,;\underbrace{(C_1\ \mathit{poss} \mid \cdots \mid C_m\ \mathit{poss})}_{\Sigma} \end{cases}}_{\gamma}$$

Again, we omit judgmental labels.

Judgmental Rules

$$\frac{}{\Gamma\,;A \Longrightarrow A\,;\cdot}\ \text{init} \qquad \frac{\Gamma, A\,;\Delta, A \Longrightarrow \gamma}{\Gamma, A\,;\Delta \Longrightarrow \gamma}\ \text{copy} \qquad \frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot}{\Gamma\,;\Delta \Longrightarrow \cdot\,;A \mid \Sigma}\ \text{promote}$$

Multiplicative Connectives

$$\frac{\Gamma\,;\Delta, A, B \Longrightarrow \gamma}{\Gamma\,;\Delta, A \otimes B \Longrightarrow \gamma}\ \otimes L \qquad \frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot \quad \Gamma\,;\Delta' \Longrightarrow B\,;\cdot}{\Gamma\,;\Delta, \Delta' \Longrightarrow A \otimes B\,;\cdot}\ \otimes R$$

$$\frac{\Gamma\,;\Delta \Longrightarrow \gamma}{\Gamma\,;\Delta, \mathbf{1} \Longrightarrow \gamma}\ \mathbf{1}L \qquad \frac{}{\Gamma\,;\cdot \Longrightarrow \mathbf{1}\,;\cdot}\ \mathbf{1}R$$

$$\frac{\Gamma\,;A \Longrightarrow \cdot\,;\Sigma \quad \Gamma\,;B \Longrightarrow \cdot\,;\Sigma}{\Gamma\,;A \boxplus B \Longrightarrow \cdot\,;\Sigma}\ \boxplus L \qquad \frac{\Gamma\,;\Delta \Longrightarrow \cdot\,;A \mid B}{\Gamma\,;\Delta \Longrightarrow A \boxplus B\,;\cdot}\ \boxplus R$$

$$\frac{}{\Gamma\,;\bot \Longrightarrow \cdot\,;\Sigma}\ \bot L \qquad \frac{\Gamma\,;\Delta \Longrightarrow \cdot\,;\cdot}{\Gamma\,;\Delta \Longrightarrow \bot\,;\cdot}\ \bot R$$

$$\frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot}{\Gamma\,;\Delta, \neg A \Longrightarrow \cdot\,;\Sigma}\ \neg L \qquad \frac{\Gamma\,;\Delta, A \Longrightarrow \cdot\,;\cdot}{\Gamma\,;\Delta \Longrightarrow \neg A\,;\cdot}\ \neg R$$

$$\frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot \quad \Gamma\,;\Delta', B \Longrightarrow \gamma}{\Gamma\,;\Delta, \Delta', A \multimap B \Longrightarrow \gamma}\ \multimap L \qquad \frac{\Gamma\,;\Delta, A \Longrightarrow B\,;\cdot}{\Gamma\,;\Delta \Longrightarrow A \multimap B\,;\cdot}\ \multimap R$$

Additive Connectives

$$\frac{\Gamma\,;\Delta, A \Longrightarrow \gamma}{\Gamma\,;\Delta, A \mathbin{\&} B \Longrightarrow \gamma}\ \&L_1 \quad \frac{\Gamma\,;\Delta, B \Longrightarrow \gamma}{\Gamma\,;\Delta, A \mathbin{\&} B \Longrightarrow \gamma}\ \&L_2 \quad \frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot \quad \Gamma\,;\Delta \Longrightarrow B\,;\cdot}{\Gamma\,;\Delta \Longrightarrow A \mathbin{\&} B\,;\cdot}\ \&R$$

$$\frac{}{\Gamma\,;\Delta \Longrightarrow \top\,;\cdot}\ \top R \quad \frac{}{\Gamma\,;\Delta, \mathbf{0} \Longrightarrow \gamma}\ \mathbf{0}L$$

$$\frac{\Gamma\,;\Delta, A \Longrightarrow \gamma \quad \Gamma\,;\Delta, B \Longrightarrow \gamma}{\Gamma\,;\Delta, A \oplus B \Longrightarrow \gamma}\ \oplus L \quad \frac{\Gamma\,;\Delta \Longrightarrow A\,;\cdot}{\Gamma\,;\Delta \Longrightarrow A \mathbin{\&} B\,;\cdot}\ \oplus R_1 \quad \frac{\Gamma\,;\Delta \Longrightarrow B\,;\cdot}{\Gamma\,;\Delta \Longrightarrow A \mathbin{\&} B\,;\cdot}\ \oplus R_2$$

Exponentials

$$\frac{\Gamma, A\,;\Delta \Longrightarrow \gamma}{\Gamma\,;\Delta, !A \Longrightarrow \gamma}\ !L \quad \frac{\Gamma\,;\cdot \Longrightarrow A\,;\cdot}{\Gamma\,;\cdot \Longrightarrow !A\,;\cdot}\ !R \quad \frac{\Gamma\,;A \Longrightarrow \cdot\,;\Sigma}{\Gamma\,;?A \Longrightarrow \cdot\,;\Sigma}\ ?L \quad \frac{\Gamma\,;\Delta \Longrightarrow \cdot\,;A}{\Gamma\,;\Delta \Longrightarrow ?A\,;\cdot}\ ?R$$