Maximum Number of Nonzero Persistence Cycles in a Vietoris-Rips Filtration

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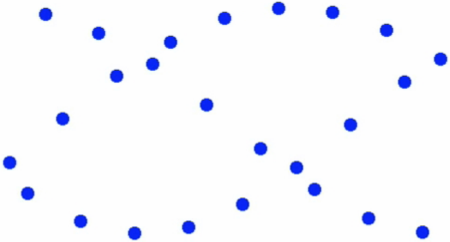
Data RTG, Duke University

July 22, 2014

Background - Persistent Homology



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- Some work done on static topological shapes, but none on filtrations.
- Bounding the number of holes is the first step.

Simplicial Approximation

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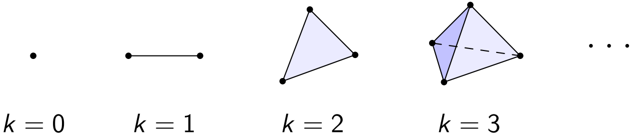
- k-simplex



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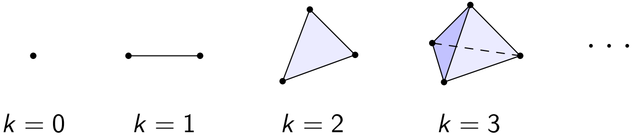

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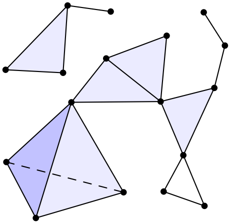

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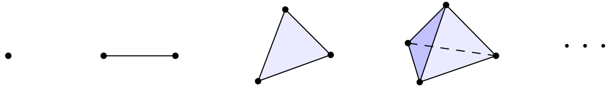

$k = 0$ $\quad$ $k = 1$ $\quad$ $k = 2$ $\quad$ $k = 3$

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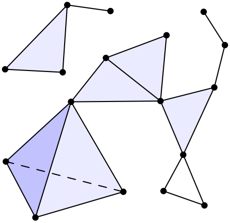

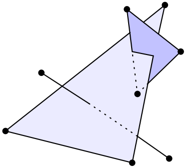

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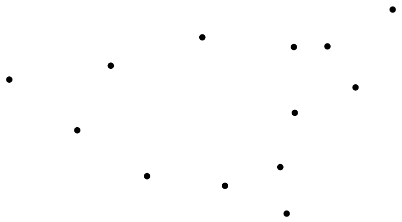

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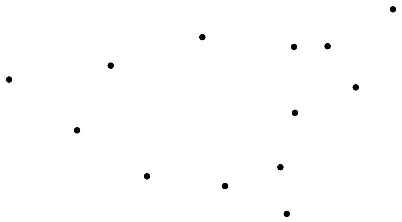

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Definition

Given a point cloud $P \subset \mathbb{R}^d$ and some $r > 0$, the *geometric graph* $G_r(P)$ is the graph with vertex set P and edge set $\{\{\mathbf{p}_i, \mathbf{p}_j\} : ||\mathbf{p}_i - \mathbf{p}_j|| \leq r\}$.

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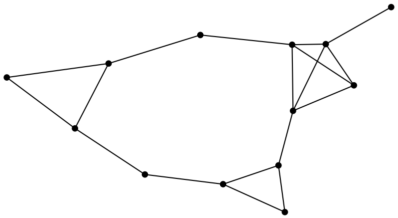

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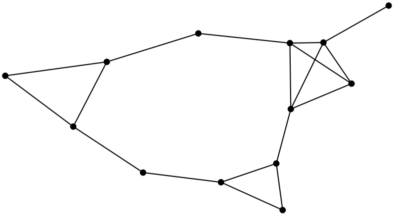

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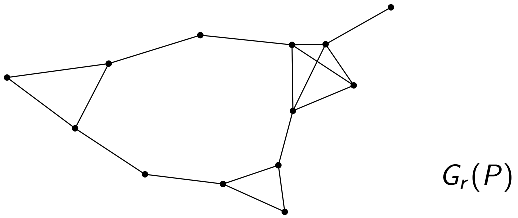

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Given a point cloud $P \in \mathbb{R}^d$ and some $r > 0$, the *Vietoris-Rips complex* $V_r(P)$ is the simplicial complex consisting of the cliques of $G_r(P)$.

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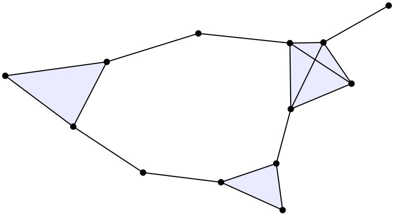

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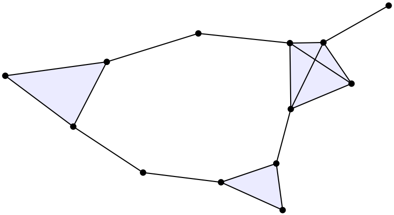

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Definition

The collection $\{V_r(P)\}_{r \in R}$, where R is the set of pairwise distances between points in P, is called the *Vietoris-Rips filtration*.

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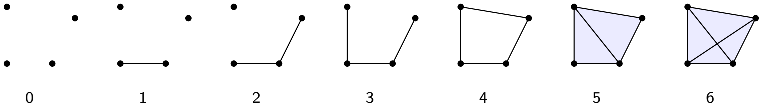

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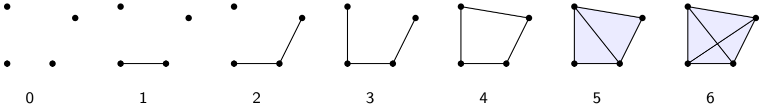

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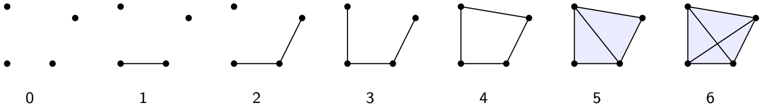

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Loosely defined, a cycle is a *nonzero persistence (NZP) cycle* if it is nontrivial upon its birth.

Positive and Negative Edges

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Let $P \subset V_{r_1}(P) \subset V_{r_2}(P) \subset \cdots \subset V_{r_m}(P)$ be a VR filtration, and let $e_1, e_2, \ldots, e_m$ be the corresponding sequence of dropped edges.

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An edge e_i is called *negative* if it connects two disjoint components in $V_{r_{i-1}}(P)$. Otherwise, it is called *positive*.

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Observe that only positive edges can birth cycles (trivial or nontrivial).

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An edge $e_i = \{u, v\}$ *births a nontrivial cycle if and only if it is positive, and* u *and* v *have no common neighbors in* $V_{r_i}(P)$.

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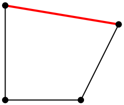

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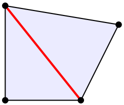

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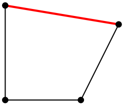

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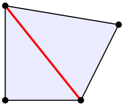

No new nontrivial cycle is born.

Thus, we can count the nonzero persistence cycles by counting the number of positive edges with no common neighbors of its endpoints.

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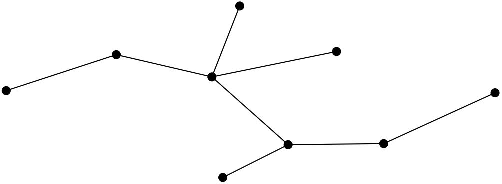

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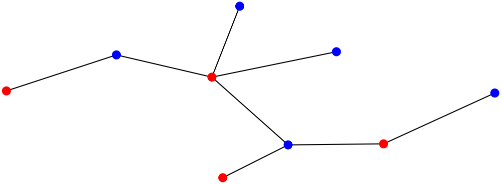

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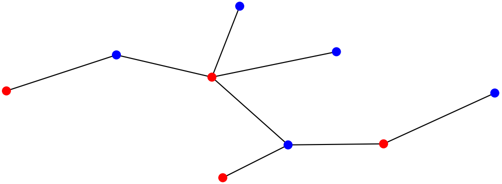

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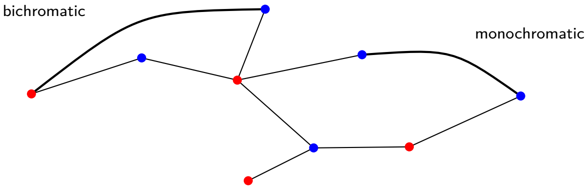

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Theorem

$$\alpha(n, F) = rb - (n - 1).$$

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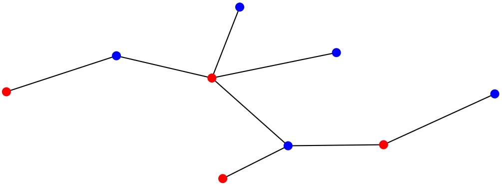

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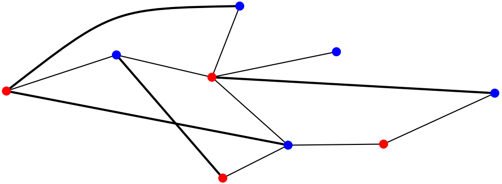

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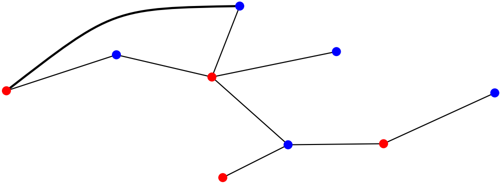

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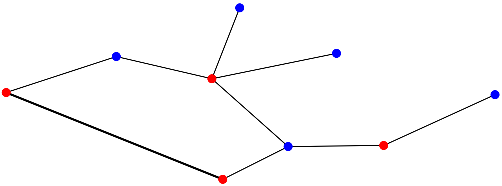

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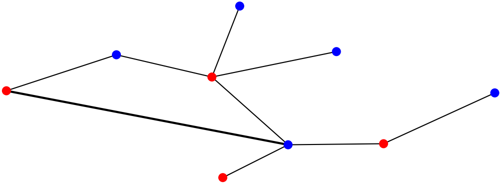

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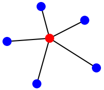
Corollary 2

If F is a star, then there are no nonzero persistence cycles.



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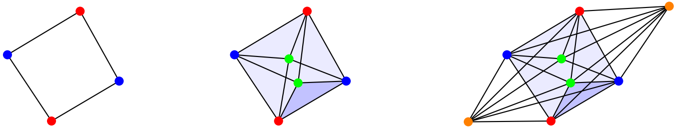

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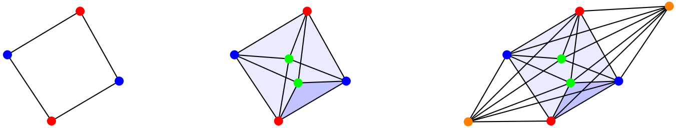

  - Possible connection to Turán's theorem: The number of edges in a graph $G \not\supseteq K^r$ with n vertices is at most $\frac{1}{2}n^2\frac{r-2}{r-1}$, and the unique maximal structure is a Turán graph.